# APPROACHES FOR 2—ECHELON MULTI ITEM DETERMINISTIC INVENTORY SYSTEM

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CERTIFICATE

This is to certify that the present work on 'Approaches for 2-Echelon, Multi-Item, Deterministic Inventory System' by Alok Chakravarti has been carried out under our supervision and has not been submitted elsewhere for the award of a degree.

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ABSTRACT

The present work deals with the single activity and multi activity inventory systems. The single activity inventory system involves consideration of multi-item, single location problems which are formulated for static and dynamic deterministic demands. For the dynamic demand model a periodic review, finite horizon inventory system is considered. The objective is to determine the economic frequency of ordering and the order quantity of each item so as to minimize the total cost of the system.

In the multi-activity inventory system two specific problems are undertaken, viz., the multi-item, single supplier--single customer problem and the single item, single supplier--**m** customer problem. For both the problems, mathematical models and solution methodologies have been developed for the integrated ordering policy as well as the individual ordering policy of the supplier and the customer. Further the following order cost structures are considered for the first problem.

i) Single order cost for all the items.

ii) Separate order cost for each item.

iii) Separate order cost for eac item alongwith a joint order cost for all the items.

The solution methodologies for integrated ordering policies are iterative in nature and yield optimal or near optimal solution. Numerical examples are given to illustrate the solution methodologies.

# CHAPTER I

# INTRODUCTION

Lot of interest centres around the inventory policy decisions at the stocking point in multi-item, multi-echelon and multi-location distribution systems. The most common notion of a multi-echelon inventory system is one involving a number of retail outlets (stores, facilities, installations, bases) which as suppliers satisfy customer demands for goods and also act as customers to higher level wholesale activities (warehouses, depots, factories). These wholesale activities in turn may act as customers to still higher level activities and this chain may go on. Figure 1.1 shows in general a multi-echelon inventory system. The chain of activities depicted are product dependent, i.e. for different products there may be different structures. In the multi-echelon systems if the network has one incoming link for each facility and no loops are formed, i.e. flow is acyclic then it is called an arborescence or inverted tree structure. In a practical situation, it is possible to have a retailer ordering from more than one wholesaler or a retailer supplying another retailer and so on. In the literature two special arborescence structures are mentioned, viz., the series structure and the parallel structure. The series structure comprises of two or

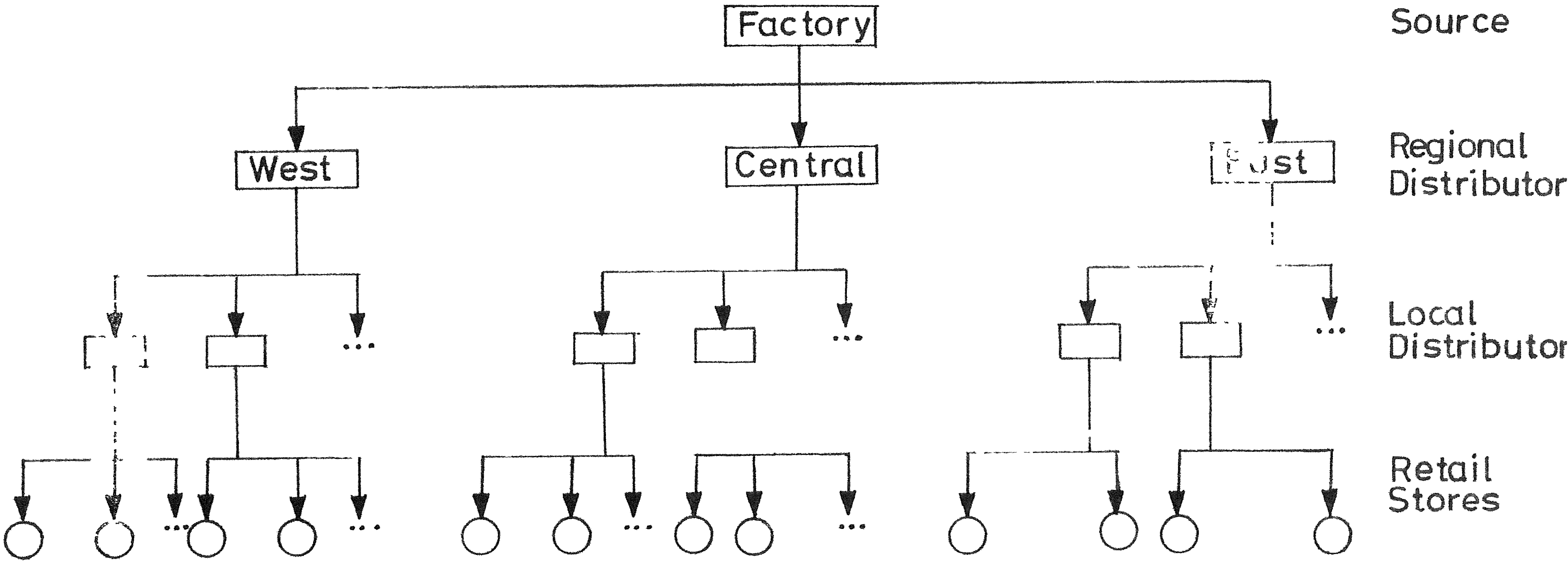



Fig. 1·1 Distribution system of arborescence structure

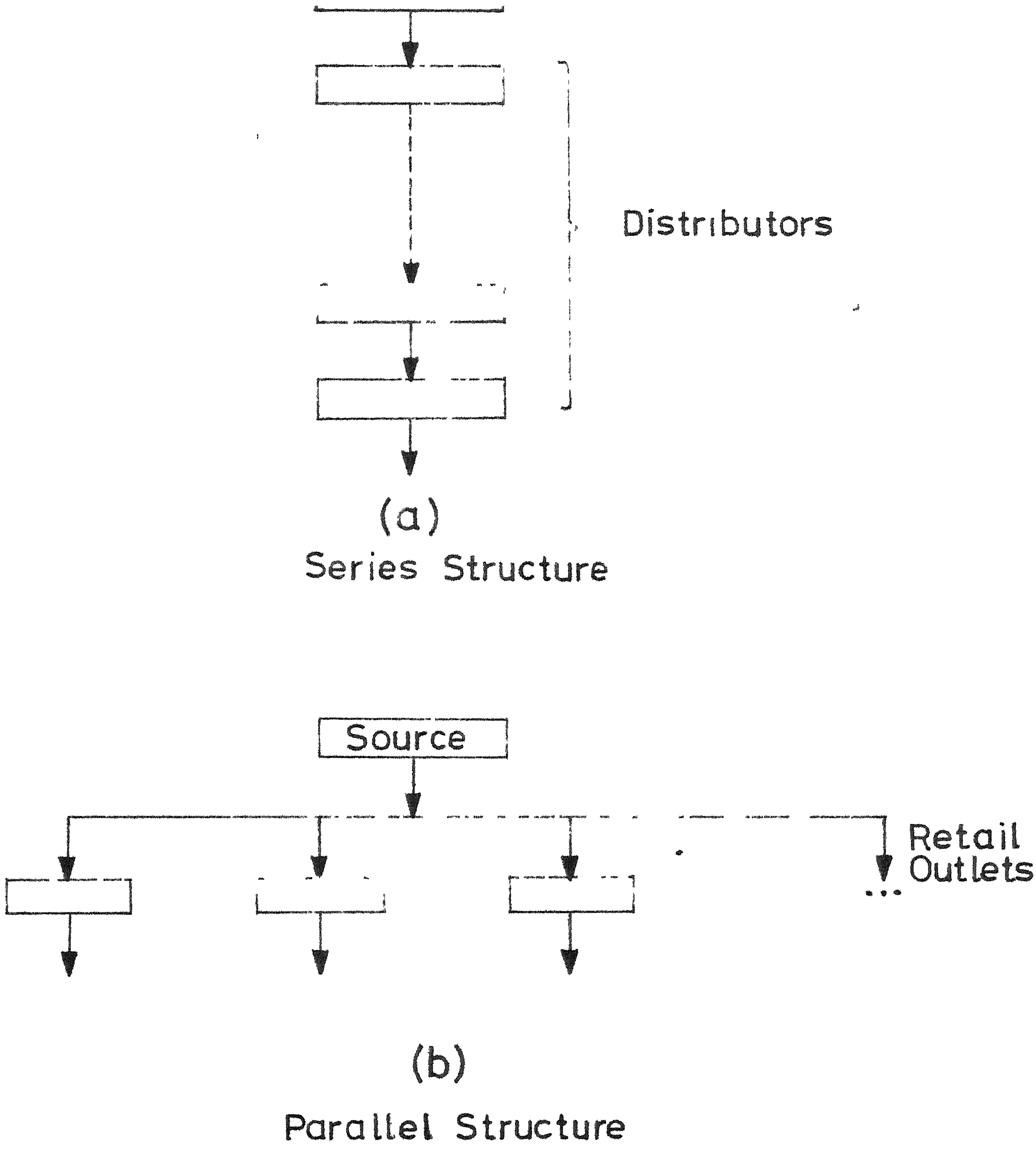



Fig. 1·2 Special arborescence structures

more activities where each activity supplies only one lower level activity. In the parallel structure a number of activities experience independent demands. All the activities are supplied from a single source. Figure 1.2 shows the two special arborescent structures. A number of practical distribution systems have parallel arborescence structures, e.g. tea distributors, chain of departmental stores supplied by a single regional warehouse, company owned service stations, etc. Till now we have been referring to the term activity without having explicitly defined what it means. In the literature this term has varied usage but the most general definition of activity is that it is an entity in which physical stock is maintained and from which demands are met as and when they occur. These entities also receive supply from some specified or implied source(s). The term activity has also been used in the context of production, where it may represent a stage of production alongwith the in-process inventory. Due to varied nomenclature the generic term activity is used only when we are not in a position to explicitly define the entity.

An inventory control problem is characterized by its features. The number of distinguishing features of an inventory problem has proliferated to such an extent that it is no longer possible to enumerate all of them. Clark [2] has given a comprehensive survey of inventory control

problem structure with special reference to multi-echelon theory. An inventory problem characterized by the environmental assumptions is optimized w.r.t. some prescribed performance characteristics by establishing certain norms or policies. Models are developed for single and multi-activity problems.

Single Activity Problem

Single activity inventory problem can be classified as single item vs. multi-item, static vs. dynamic, deterministic vs. stochastic demand, etc. Exhaustive literature exists on single item, single location problem. Researchers like Shu [9], Goyal [3] and others have presented iterative procedures for obtaining near optimal solutions to multi-item, single location inventory problem with static deterministic demand. They did not consider the effect of shortages. Scant attention seems to have been paid to multi-item case with backlogging or with dynamic deterministic demand. With a view to fill in this void, the first phase of this research work deals with multi-item, single location inventory problem having static and dynamic demands. Models with and without backlog and lead time are considered. It is shown that for a policy to be optimal all the items must be jointly replenished.

Multi Activity Problem

Multi activity inventory problems are basically of two types, viz., single-echelon and multi-echelon. Single-echelon, multi activity problems are not of much significance. Multi-echelon, multi activity problems had till recently been treated independently at the various levels by different authors. Only recently Schwarz [8] and Goyal [4] have adopted a systems approach in the development of solution procedures for a 2-echelon inventory system comprising of a single supplier and a single customer. They have obtained optimal solutions for the single item case. Goyal [4] has shown that the aggregate approach to such problems results in considerable savings as compared to an independent ordering policy followed by the supplier and the customer. The contribution of these two authors has provided the necessary motivation for the present work. The following specific problems are undertaken in this work.

1. Multi-item, single supplier - single customer inventory problem.
2. Single supplier - m customer inventory problem.

1.1 Organization of the Thesis:

In the following chapters the mathematical formulations and solution methodologies of the three problems concerning single activity and multi-activity inventory systems mentioned

in the previous section are presented alongwith the relevant literature. Chapter 2 deals with the single location, multi-item problem considering both the static and dynamic cases for deterministic demand. In Chapters 3 and 4 the problems of multi-item, single supplier - single customer and single item, single supplier - m customers are presented. The final conclusions and the avenues for further research work form the text of Chapter 5.

# CHAPTER II

## MULTI ITEM, SINGLE LOCATION INVENTORY PROBLEM

In this chapter the mathematical modeling and the solution procedure for the multi-item, single location inventory problem are presented. The chapter comprises of two major sections dealing with the static and dynamic deterministic demands. Each of these problem situations are considered with and without backlogging as well as lead time. Before presenting the problems the relevant literature surveyed is given in the next section.

### 2.1 Literature Review

Starr and Miller [12] were probably the first to introduce the concept of constrained multi-product inventory systems. They assumed instantaneous replenishment and backorders were not permitted. They obtained optimal stocking policy with the help of Lagrangian multipliers. Naddor [6] considered a two product system and proposed two optimal policies. The first policy pertained to the joint ordering of products using classical lot size formula. In the second policy independent ordering of the items was permitted.

Shu [9] has described a method for determining the replenishment frequencies of 2 items involving joint

replenishment. The model assumes that the item with the highest demand is the most frequently ordered item. Further the frequency of replenishment of any item is an integer multiple of the frequency of replenishment of the most frequently ordered item. Nocturne [7] has contradicted Shu's argument and has illustrated with the help of an example that the item with higher demand need not necessarily be the item more frequently ordered. Goyal [3] has presented a simple method for implementing joint replenishment systems. The procedure is iterative in nature. He has provided graphs and tables as aids to facilitate the calculation of the optimal frequency of replenishment of any item. Silver [10,11] has proposed a simpler non-iterative approach for coordinated replenishment of a group of items. It needs to be pointed out that the approaches suggested by Goyal and Silver do not guarantee the optimality of the solution. Wagner-Whitin [13] have developed a dynamic programming approach to obtain optimal solution to single item, single location problem, having dynamic deterministic demand.

Keeping in view the existing literature, the models pertaining to the multi-item, single location problem are extended.

## 2.2 Problem Statement

Consider a store which receives its supply of products from a single supplier which is assumed to have infinite amount of all the products. The store experiences a static deterministic demand. The order cost for the store for all the items remains fixed irrespective of the size of the order. The problem is to determine the economic frequency of ordering and the order quantity of each item. The various cases dealt with are.

1. Zero lead time with no shortages.
2. Zero lead time with backlogging permitted.
3. Finite lead time with constant and variable lead times for all the items.

### 2.2.1 Assumptions

Various assumptions involved in the development of the models are as follows:

1. Demand rate of each item is deterministic and static.
2. Replenishments cannot be split, i.e. the entire order must be delivered at a time.
3. Quantity discounts are not permitted.
4. Items once ordered or on hand at the store cannot be returned to supplier or disposed off.
5. There is one and only one source of supply for all the products. Capacity of source is infinite.

### 2.2.2 Nomenclature

The following notations have been used in the present study,

$S_o$ - order cost for all the items.

m - number of items.

T - time interval between two consecutive orders in years.

$\tau$ - constant component of lead time.

For the j-th Item

$D_j$ - demand per year.

$h_j$ - holding cost per unit per unit time.

$P_j$ - purchase cost

$b_j$ - backlogged quantity

$Q_j$ - order quantity

$K_j$ - backorder cost per unit per unit time.

$\tau_j$ - variable component of lead time.

### 2.2.3 Problem Formulation

Case I: Zero Lead Time with no Shortages:

First we state an important theorem, which forms the basis of our problem formulation.

Theorem: For single order cost structure, it is always optimal to have joint replenishment of all the items.

Proof: Let the m items be replenished at different times and let k be the most frequently ordered item such that

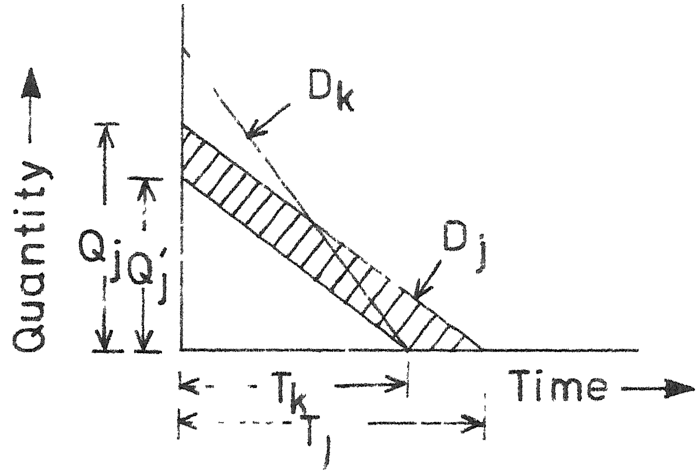



Fig.2·1 Structure of inventory holding cost with and without joint replishment

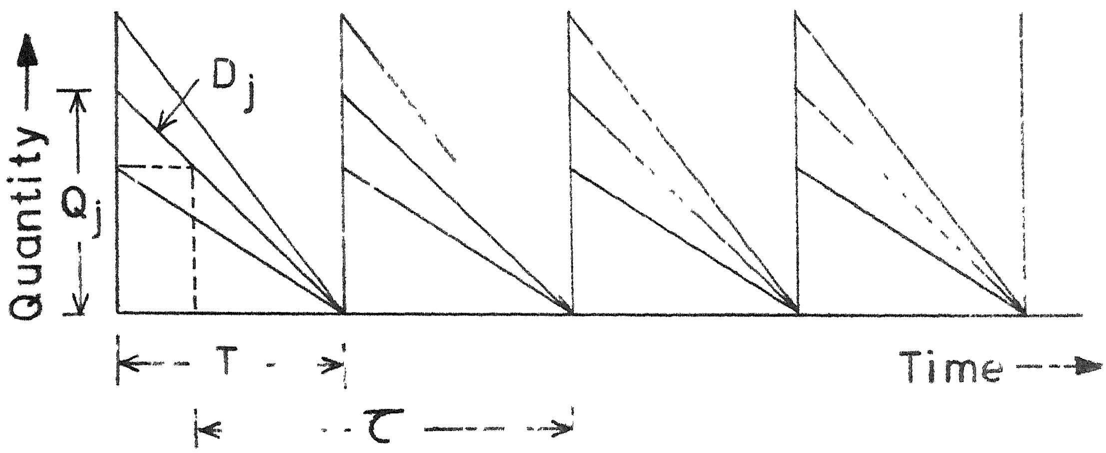



Fig.2·2 Inventory picture with joint replishment of all items

$T_k \leq T_i$, $i = 1, \ldots, m$. Let $j$ be any other item such that $T_j > T_k$, $j = 1, \ldots, m$, $j \neq k$. Then the j-th item will have a positive inventory at time $T_k$. If the quantity of the j-th item ordered is so adjusted that inventory at time $T_k$ is zero, then there is a saving in the holding cost equal to the shaded area shown in figure 2.1. The joint replenishment also results in the reduction of the ordering cost. Extending the same logic for all items one concludes that the optimal ordering policy is to jointly replenish all the items.

Since the demand is static and deterministic and from the above we know that it is always optimal to jointly replenish all the items, the resulting inventory system is shown in figure 2.2. From the figure it is evident that the quantity of each item ordered depends upon the time interval between orders for the entire group and the individual demand rate.

Mathematical Model and Solution Methodology

Total annual cost = (Purchase Cost) + (Order Cost) + (Holding Cost)

$$C(T) = \sum_{j=1}^{m} D_j P_j + \frac{S_o}{T} + \frac{1}{2} T \sum_{j=1}^{m} h_j D_j \tag{2.1}$$

The variable cost per year, $V(C(T))$, is given by

$$V(C(T)) = \frac{S_o}{T} + \left( \sum_{j=1}^{m} \frac{h_j D_j}{2} \right) T$$

Since T is a continuous variable, the optimal value of T can be obtained by solving $\frac{\partial V(C(T))}{\partial T} = 0$. Thus the most economical ordering policy is to place orders at intervals of $T^*$, where

$$T^* = \left( \frac{2 S_0}{\sum_{j=1}^{m} h_j D_j} \right)^{\frac{1}{2}} \tag{2.2}$$

The economic order quantity of the j-th item is given by,

$$Q_j^* = D_j T^* \tag{2.3}$$

Case II: Backlogging is Permitted:

Before presenting the mathematical model we enunciate another theorem and also give its proof.

Theorem:

For an inventory system with backlogging permitted and having a single order cost structure, it is always optimal to jointly replenish all the items.

Proof: Let 'm' items be replenished at different times and $T_k$ be the smallest cycle time corresponding to the k-th item such that, $T_k \leq T_i$, $i = 1, \ldots, m$. Let $T_j$, $T_j > T_k$, be the cycle time of any other item. There are two possible cases. The item j depletes to zero level at a time a) less than $T_k$, b) greater than $T_k$. Both these cases are depicted in Fig. 2.3. In both the cases, reducing $T_j$ to $T_j' = T_k$, the various possible ordering policies for item j are:

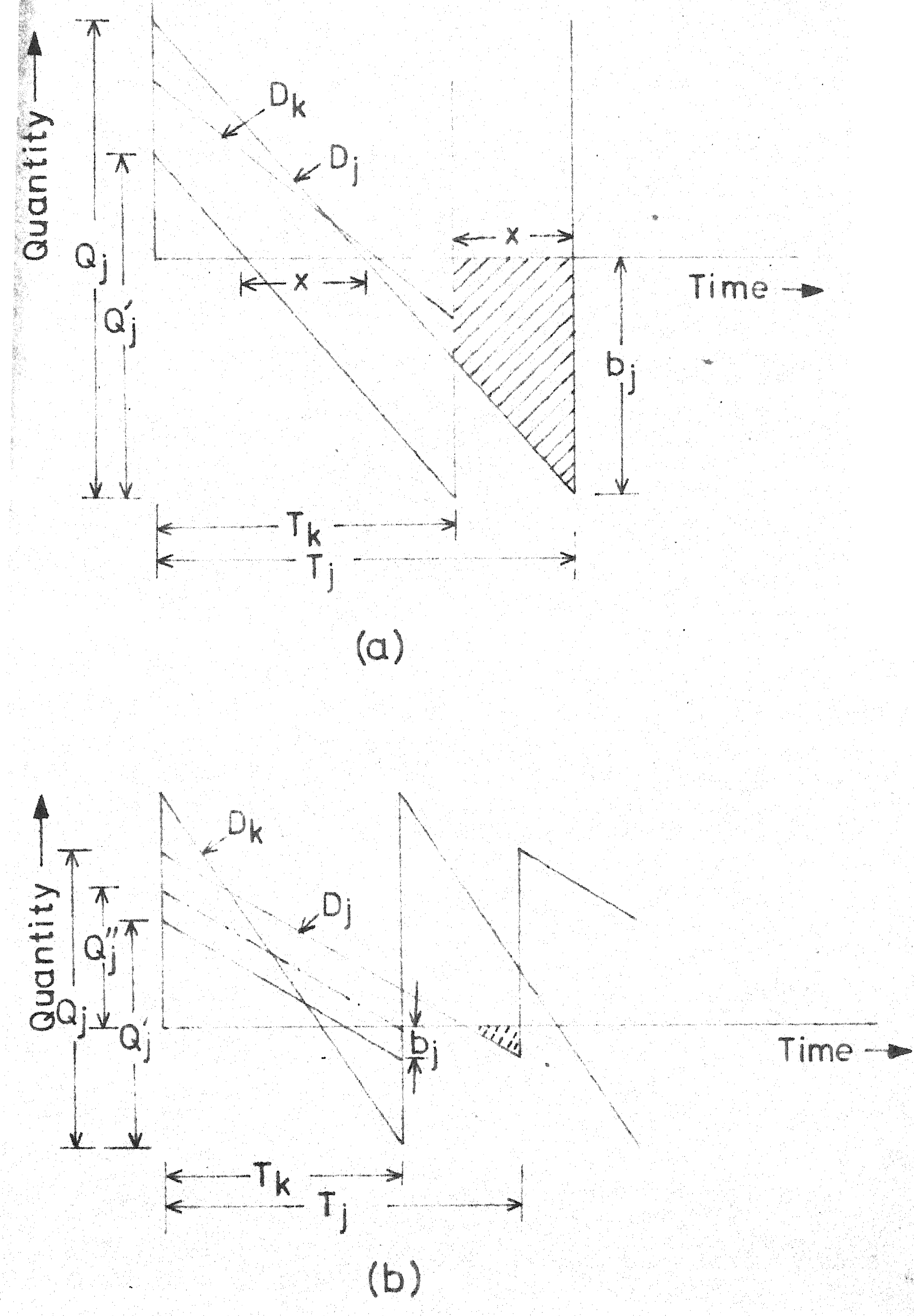



Fig. 2·3 Representation of two cases with backlogging

i) Order $Q_j$, i.e., do not change the order quantity of the j-th item. This is possible only if the j-th item corresponds to the configuration of figure 2.3(a). This ordering policy results in savings in the backorder cost as shown by the hatched area. Even if the j-th item follows the configuration shown in Figure 2.3(b), savings accrue in the backorder cost as shown by the shaded area. But it is obvious that a better policy is to reduce order quantity to atleast $Q_j''$ as then inventory held at time $T_k$ is zero.

ii) Order $Q_j'$, i.e., reduce the order quantity from $Q_j$ to $Q_j'$, where $Q_j'$ is the order quantity obtained by shifting the demand rate line by x, $x = (T_j - T_k)$. For this policy backorder cost remains the same, while the holding cost is reduced as shown in Figure 2.3.

iii) The order quantity is so adjusted that the net order quantity lies between $(Q_j - b_j)$ and $(Q_j' - b_j)$. For this policy there is savings in both the holding and backorder costs.

Thus savings in holding and backorder costs occur in all the three ordering policies apart from the savings in order cost by making joint replenishment of items k and j. Extending this logic for all the items, one can state that it is always optimal to replenish all the items jointly.

Mathematical Model and Solution Methodology.

Total cost per year, $C(T, b_j)$ =

$$= \text{(Purchase Cost)} + \text{(Order Cost)} + \text{(Holding Cost)} + \text{(Backorder Cost)}$$

$$= \sum_{j=1}^{m} D_j P_j + \frac{S_o}{T} + \sum_{j=1}^{n} \frac{h_j (Q_j - b_j)^2}{2Q_j} + \sum_{j=1}^{m} \frac{K_j b_j^2}{2Q_j} \tag{2.4}$$

Substituting, $\frac{Q_j}{D_j} = T, \quad j = 1, \ldots, n$, we get,

$$C(T, b_j) = \sum_{j=1}^{n} D_j P_j + \frac{S_o}{T} + \sum_{j=1}^{m} \frac{h_j (D_j T - b_j)^2}{2D_j T} + \sum_{j=1}^{n} \frac{K_j b_j^2}{2D_j T}$$

$$= \sum_{j=1}^{n} D_j P_j + \frac{S_o}{T} + \sum_{j=1}^{n} \frac{(D_j^2 T^2 + b_j^2 - 2D_j T b_j)}{2D_j T} + \sum_{j=1}^{m} \frac{K_j b_j^2}{2D_j T} \tag{2.5}$$

Taking partial derivative of $C(T, b_j)$, w.r.t. $b_j$ and $T$ and equating to zero, we get,

$$T^2 = \left[ \frac{2S_o + \sum_{j=1}^{m} \frac{b_j^2 (h_j + K_j)}{D_j}}{\sum_{j=1}^{n} h_j D_j} \right], \text{ and} \tag{2.6}$$

$$T = \frac{(K_j + h_j)\, b_j}{D_j h_j}, \quad j = 1, \ldots, m \tag{2.7}$$

There are (m+1) unknown variables and (m+1) equations. Thus we can solve these (m+1) equations to obtain the optimal values of $b_j^*$ and $T^*$. Substituting for $b_j$ from eq.(2.7) into eq.(2.6), we get,

$$T = \left[ \frac{2S_o}{\sum_{j=1}^{m} \left( \frac{K_j h_j . D_j}{K_j + h_j} \right)} \right]^{\frac{1}{2}}, \quad \text{and} \tag{2.8}$$

substituting eq. (2.8) back into eq. (2.7), we get,

$$b_j^* = \left[ \frac{D_j \; h_j}{(K_j + h_j)} \right] \left[ \frac{2S_o}{\sum_{j=1}^{m} \left( \frac{K_j h_j . D_j}{K_j + h_j} \right)} \right]^{\frac{1}{2}} \tag{2.9}$$

Knowing $T^*$ the optimal order quantity $Q_j^*$ for any item $j$ can be calculated as

$$Q_j^* = D_j \, T^* \tag{2.10}$$

Case III: Finite Lead Time

a) Constant lead time for all items: If there is a constant lead time for all the items and they are jointly replenished then all the items will have the same reorder point. The only difference in this case from the previous cases is that the reorder point shifts.

Now for lead time equal to $\tau$, we have

$$A = \left\lceil \frac{\tau}{T} \right\rceil \tag{2.11}$$

where A is an integer.

Reorder point on quantity axis,

$$r_h = (\tau - AT)\ D \tag{2.12}$$

where D is the demand of any item. Fig. 2.2 shows the reorder point.

b) Variable lead Time: If the lead time for all items is different then to obtain the advantage of joint replenishment the reorder points for all items must be different. The reorder point for any item j can be calculated using the following relationship.

$$A_j = \left\lceil \frac{\tau_j}{T} \right\rceil \tag{2.13}$$

where $A_j$ is an integer.

For the j-th item, the reorder point on quantity axis is given by,

$$r_{h_j} = (\tau_j - A_j T)\ D_j \tag{2.14}$$

2.3 Problem Statement

Consider a store which receives its supply of products from a single supplier having infinite amount of all the items. The demand at the store is dynamic and deterministic in nature. The problem is to determine the regeneration points and order quantities, for a finite horizon, periodic review inventory system characterized by the features given above. The various cases dealt with are characterized by zero and finite lead time. Shortages are not permitted.

### 2.3.1 Assumptions:

Various assumptions necessary for the development of the models are as follows:

1. Demand rate is dynamic and deterministic for each item.
2. Replenishments cannot be split.
3. Quantity discounts are not permitted.
4. Costs are linear or concave in nature.
5. Planning horizon is finite.
6. Periodic reviews are made and the demand rate is uniform in a period.
7. Inventory holding cost is charged on the inventory held at the end of the period. No cost is incurred on the items demanded during a period.
8. Shortages are not allowed.

### 2.3.2 Nomenclature

The following notations have been used in the present study.

$N$ – number of periods in planning horizon.

$M_{jkn}$ – total cost of procuring in period (k+1) to meet demands of period (k+1) through n, where $k = 0,1,\ldots,n-1$, and $k+1 \leq n \leq N$ of the j-th item.

$x_{jt}$ amount of j-th item procured in period 't'.

$I_{j(t-1)}$ – inventory of j-th item at the end of period (t-1).

$C_{j(k+1)}$ – variable order cost component. Cost of ordering in period (k+1) to meet demand of period (k+1) through n of j-th item.

$f_n$ - optimal cost of the system for n periods.

$h_{j(k)}$ holding cost of j-th item in period k.

m - number of items

$r_h$ reorder point to obtain the replenishment at the beginning of h-th period.

$D_{jk}$ - Demand of the j-th item in k-th period.

$D_k$ - demand in k-th period of any item.

$r_{h_j}$ - reorder point to obtain replenishment at the beginning of h-th period for the j-th item.

2.3.3 Problem Formulation.

Case I : Zero Lead Time, No Shortages:

To solve this problem we resort to a dynamic programming approach similar to the Wagner - Whitin algorithm used for single item, dynamic demand inventory problem. A recursive formula is developed which when applied yields the optimal solution at any stage.

Mathematical Model and Solution Methodology:

The total cost of procuring in period (k+1) to meet demands of periods (k+1) through n, $M_{jkn}$, is given by

$$M_{jkn} = \begin{cases} C_{j(k+1)}\,(D_{j(k+1)}) & \text{for } n = k+1 \\ C_{j(k+1)}\,[D_{j(k+1)} + D_{j(k+2)} + \dots + D_{jn}] & \\ \quad + h_{j(k+1)}\,[D_{j(k+2)} + D_{j(k+3)} + \dots D_{jn}] + \dots h_{j(n-1)}\,(D_{jn}) & \text{for } n > k+1 \end{cases} \qquad (2.15)$$

Then the recursion formula is

$$f_n = \min_{k=0,1,..,n-1} [f_k + \sum_{j=1}^{m} (M_{jkn})] \tag{2.16}$$

for $n = 1,2,\ldots, N$.

The optimality of this recursion is guaranteed by a theorem given below, which states that there always exists an optimal policy for which it is optimal to jointly replenish all the items.

Theorem: For an optimal policy all the items must satisfy the following equality.

$$(\sum_{j=1}^{m} I_{j(t-1)}) (\sum_{j=1}^{m} x_{jt}) = 0 \quad \text{for all } t \tag{2.17}$$

Proof: Suppose for some optimal policy, we have a period t such that

$$(\sum_{j=1}^{m} I_{j(t-1)}) (\sum_{j=1}^{m} x_{jt}) > 0 \tag{2.18}$$

Let p be the last period in which an order was placed. Perturb (2.18) by increasing and decreasing $x_{jt}$ and decreasing and increasing $x_{jp}$ respectively. This perturbation causes a change in the total cost, and this is given by (2.14) and (2.15).

$$\text{Perturbed cost} = \sum_{j=1}^{m} \Big[C_{jp} (x_{jp}-\delta_j) + C_{jt}(x_{jt}+\delta_j) + \sum_{k=p}^{t-1} h_{jk} (I_{jk}-\delta_j)\Big] \tag{2.19}$$

$$\text{or Perturbed cost} = \sum_{j=1}^{m} \Big[C_{jp}(x_{jp}+\delta_j) + C_{jt} (x_{jt}-\delta_j) + \sum_{k=p}^{t-1} h_{jk} (I_{jk} + \delta_j)\Big] \tag{2.20}$$

If the costs are concave, then we know

$$\sum_{j=1}^{m} \left[ C_{jp} (x_{jp}) + C_{jt} (X_{jt}) + \sum_{k=p}^{t-1} h_{jk} (I_{jk}) \right] \geq$$

$$\sum_{j=1}^{m} \Bigg[ \frac{1}{2} [C_{jp}(x_{jp}+\delta_j) + C_{jp}(x_{jp}-\delta_j)] + \frac{1}{2} [C_{jt}(x_{jt}+\delta_j) + C_{jt}(x_{jt}-\delta_j)] + \frac{1}{2} \left[ \sum_{k=p}^{t-1} h_{jk} (I_{jk}+\delta_j) + \sum_{k=p}^{t-1} h_{jk} (I_{jk} - \delta_j) \right] \Bigg]$$

$$= \frac{1}{2} \sum_{j=1}^{m} \left[ C_{jp} (x_{jp}-\delta_j) + C_{jt}(x_{jt}+\delta_j) + \sum_{k=p}^{t-1} h_{jk} (I_{jk}-\delta_j) \right] + \frac{1}{2} \sum_{j=1}^{m} \left[ C_{jp}(x_{jp}+\delta_j) + C_{jt} (x_{jt}-\delta_j) + \sum_{k=p}^{t-1} h_{jk} (I_{jk} + \delta_j) \right] \tag{2.21}$$

The R.H.S. of (2.21) represents the perturbed costs. Thus we see that one of the perturbed costs is atleast as good as the original cost. Hence there is always an optimal policy which has,

$$\left( \sum_{j=1}^{m} I_{j(t-1)} \right) \left( \sum_{j=1}^{n} x_{jt} \right) = 0$$

Case II : Finite Lead Time:

a) Constant Lead Time for all the Items: For a constant lead time for all the items only the reorder point is different and rest of the model remains the same. The reorder point can be determined as follows. Let,

$$A = \lceil \tau \times N \rceil \tag{2.22}$$

where A is an integer. The reorder point, $r_h$, is given by,

$$r_h = (\tau - A/N)\, D_{(h-(A+1))} + \sum_{k=h-A}^{p} D_k \tag{2.23}$$

where p is the next regeneration point.

b) Variable Lead Time: If all the items have different lead times then to have coordinated replenishment at any period the reorder times and reorder quantities for all items must be different. For any given lead time $\tau_j$ for the j-th item, we have,

$$A_j = \lceil \tau_j N \rceil \tag{2.24}$$

where $A_j$ is an integer. The reorder point, $r_{h_j}$, is given by,

$$r_{h_j} = (\tau_j - A_j/N)\, D_{j(h-(A_j+1))} + \sum_{k=h-A_j}^{p} D_{jk} \tag{2.25}$$

where p is the next regeneration point.

## 2.4 Epilogue:

In this chapter on single location, multi-item, inventory system a number of extensions of the existing work on similar topics have been presented. Methods to obtain optimal ordering policy for each case have been provided alongwith the proofs of the results used in the development of the models. The next chapter deals with a 2-echelon inventory system comprising of a single supplier and a single customer.

# CHAPTER III

## SINGLE SUPPLIER - SINGLE CUSTOMER, MULTI-ITEM INVENTORY SYSTEM

In this chapter we present the mathematical modeling, solution procedures and numerical examples to illustrate the solution procedures and the savings accruing under each policy, for the multi-item, single supplier - single customer inventory problem. Number of order cost structures have been considered for integrated and individual ordering policy of the supplier and the customer. In the end the results are discussed alongwith the special features of the proposed models. Before presenting the various models the relevant literature surveyed is provided in the next section.

### 3.1 Literature Review

In addition to the work done on single location inventory system for multiple items some researchers have also considered the problem of 2-echelon inventory system. The pioneering work in this field has been done by Schwarz [8]. He has examined the one warehouse, N-retailer deterministic inventory system. The objective is to determine the stocking policy which minimizes average system cost per unit time over an infinite time horizon. He has obtained an optimal solution for the one warehouse - one retailer problem.

Goyal [4] has developed a procedure for obtaining the optimal solution to single supplier-single customer problem for single item. He has also shown that savings accrue from an integrated ordering policy as compared to an independent ordering policy of the supplier and the customer. Other important contributions in the area of multi-echelon inventory system have come from Gross [5], Allen [1] and Zangwill [14].

## 3.2 Statement of the Problem

Consider a special type of 2-echelon inventory system comprising of only one supplier and one customer dealing with multiple items. The external demand at the customer stocking point remains uniform and deterministic for each item. The problem is of developing an inventory system having no stock-outs and minimizing the variable cost per year. For this problem we have considered a number of order cost structures and each order cost structure is treated as a different case. The various cases are.

1. Single order cost for all items, i.e. only a fixed cost is incurred for each order placed irrespective of the types of items ordered.
2. Separate order cost for each item, i.e., each item functions independent of all the other items.
3. Separate order cost for each item alongwith a fixed joint order cost.

### 3.3 Assumptions:

The system under consideration is characterized by the following features:

1. The demand rate of each item is deterministic and static.
2. Shortages are not allowed.
3. Replenishments cannot be split, i.e. the entire order quantity must be delivered at a time.
4. Quantity discounts are not permitted.
5. Holding cost is linear in nature.
6. Lead time is zero for both the supplier and the customer. The supplier can deliver goods to customer instantaneously even if the supplier itself has just received the supply.
7. Items once ordered or in hand at a location cannot be returned to source or disposed off.

### 3.4 Nomenclature:

Given below are the notations common to all the three cases discussed in the following sections. Notations relevant to any particular case are explained wherever they occur.

$m$ number of items

For the customer

$D_j$ demand per year of the j-th item.

$h_{jj}$ stock holding cost per unit per year of j-th item

$S_j$ separate order cost component for j-th item.

$S_o$ single or joint order cost component

$t_j$ time interval between successive orders of the j-th item

N number of orders placed in a year

$Q_j$ inventory of the j-th item

For the supplier:

$h_{jo}$ stock holding cost per unit per year of j-th item.

$M_j$ separate order cost component for j-th item.

$M_o$ single or joint order cost component.

$T_j$ time interval between successive orders of the j-th item.

3.5 Mathematical Model and Solution Methodologies

Case I : Single Order Cost for all Items:

In this case we consider that a fixed order cost is incurred irrespective of the types of items ordered. For such an order cost structure it is always optimal to have joint replenishment of all the items as proved in Section 2.2.3.

Individual Ordering Policy:

When the supplier and the customer operate independently the variable cost per year for the customer, V(C(t)), is, expressed as

$$V(C(t)) = \frac{S_o}{t} + \sum_{j=1}^{m} \left( \frac{D_j h_{j1} t}{2} \right) = \frac{S_o}{t} + \left( \sum_{j=1}^{m} D_j h_{j1} \right) \frac{t}{2}$$

where t is the time interval between successive orders of the customer. Since t is a continuous variable, the optimal value of t can be obtained by solving $dV/dt = 0$. Thus for the customer the most economical ordering policy is to place orders at intervals of $t^*$, where,

$$t^* = \left[ \frac{2S_o}{\sum_{j=1}^{m} D_j h_{j1}} \right]^{\frac{1}{2}} \tag{3.1}$$

This results in a variable cost of

$$V(C(t^*)) = \left( 2S_o \left( \sum_{j=1}^{m} D_j h_{j1} \right) \right)^{\frac{1}{2}} \tag{3.2}$$

The supplier receives orders at intervals of $t^*$ and his variable cost per year, $V(S(Kt^*))$, becomes

$$V(S(Kt^*)) = \frac{M_o}{Kt^*} + \frac{(K-1)\ t^*}{2} \left( \sum_{j=1}^{m} D_j h_{jo} \right) \tag{3.3}$$

where K is a positive integer and

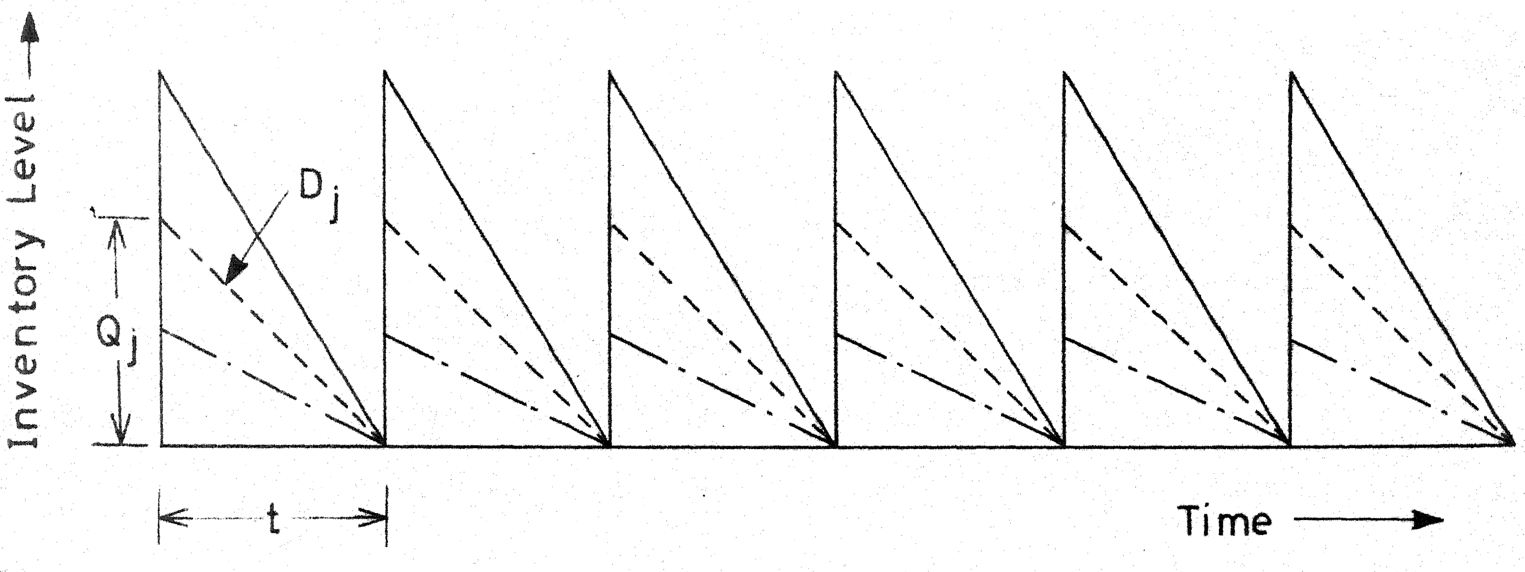
K=1 implies that supplier replenishes every customer cycle

K=2 implies that supplier replenishes every alternate customer cy

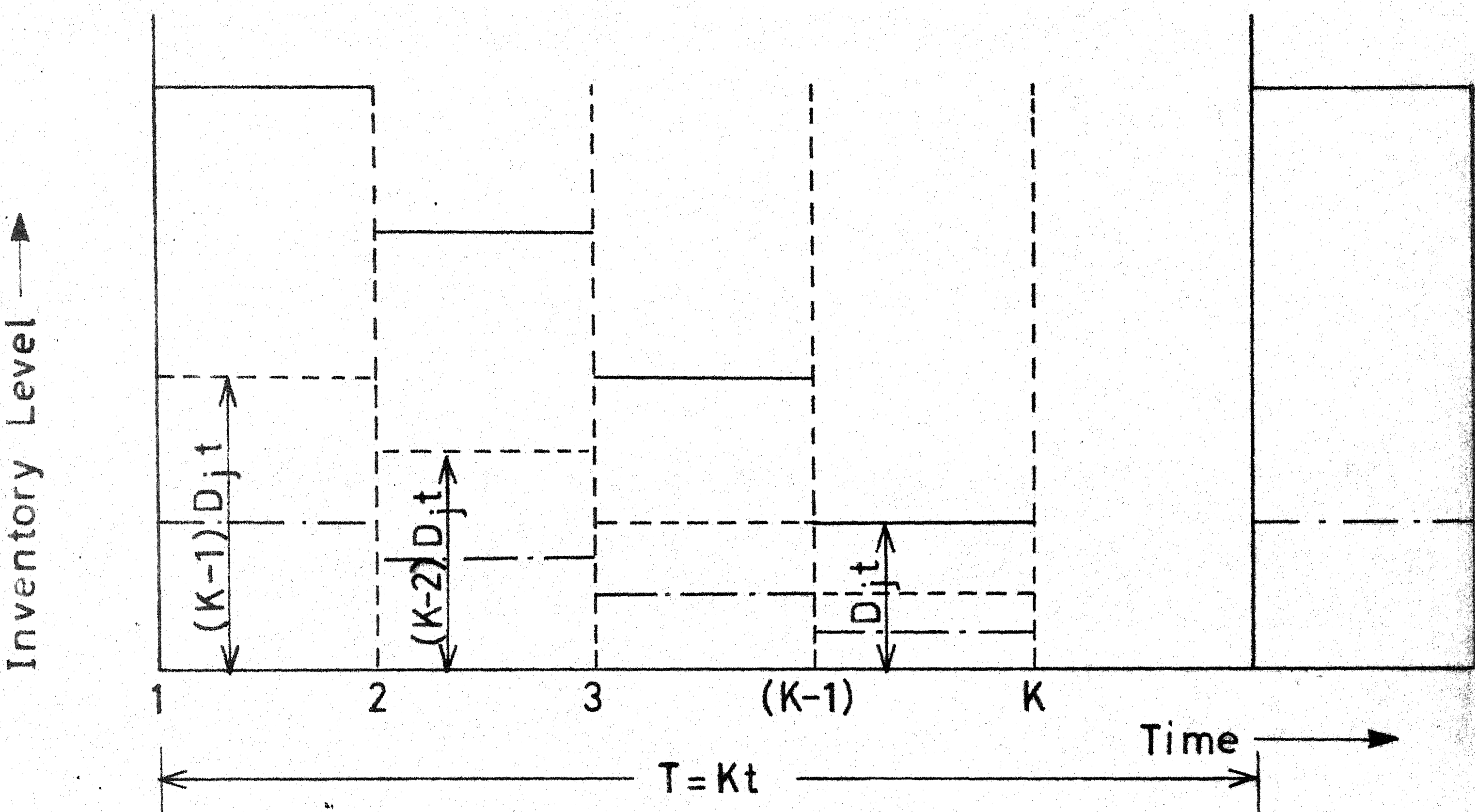
K=3 implies that supplier replenishes every third customer cycle

and so on.

For the supplier the optimum value of $K = K^*$ would be such that,


Inventory Level
D j
Q j
t
Time

(a)


Inventory Level
(K-1).D j.t
(K-2).D j.t
D j.t
1
2
3
(K-1)
K
Time
T=Kt

(b)

$$K^* (K^* + 1) > \frac{2M_o}{\left(\left(\sum_{j=1}^{m} D_j h_{jo}\right) t^{*2}\right.} \geq K^* (K^* - 1) \qquad (3.4)$$

$K^*$ can be computed from inequality (3.4) and substituted in (3.3) to obtain the variable cost per year of the supplier. The inventory picture at both the customer and supplier stocking points are shown in Fig. 3.1.

Integrated Ordering Policy:

If the supplier and the customer operate jointly the variable cost per year, $V(C(t), S(Kt))$, for the integrated ordering policy is given by

$$V(C(t), S(Kt)) = (S_o + \frac{M_o}{K}) \frac{1}{t} + \frac{t}{2} [ \sum_{j=1}^{m} (h_{j1} + (K-1)h_{jo}) D_j] \qquad (3.5)$$

Since t is a continuous variable, the optimal time interval of the integrated system is obtained by equating the first derivative to zero. Solving for t yields,

$$t^* = [2(S_o + M_o/K)/( \sum_{j=1}^{m} (h_{j1} + (K-1) h_{jo}) D_j)]^{\frac{1}{2}} \qquad (3.6)$$

Substituting the value of $t^*$ from Eq. (3.6) in (3.5), we have the variable cost per year of the integrated system as,

$$V(C(t^*), S(Kt^*)) = [2( \sum_{j=1}^{m} (h_{j1} + (K-1)h_{jo})D_j) (S_o+M_o/K)]^{\frac{1}{2}} \qquad (3.7)$$

In (3.7) only K is unknown. K can take various values, viz., K = 1,2,3,.... We select that value of $K = K^*$ which

minimizes Eq. (3.7). Substituting in (3.6) we obtain the optimal $t^*$ .

Now let us optimize the variable cost per year w.r.t. N instead of t. Here N is the ordering frequency of the customer and carries the restriction that it must be an integer. In terms of N, Eq. (3.5) can be rewritten as,

$$V(C(N),\ S(N/K)) = (S_o + \frac{M_o}{K})\ N + \frac{1}{2N}\ (\sum_{j=1}^{m}\ (D_j\ h_{j1} + (K-1)\ D_j h_{jo}))$$

A particular value of $K = K^*$ is chosen to satisfy the following inequalities,

$$V(C(N),\ S(N/K^*)) < V(C(N)\ ,\ S(N/(K^*+1)))$$

and $$V(C(N),\ S(N/K^*)) < V(C(N),\ \ S(N/(K^*-1)))$$

which on simplification yield

$$\frac{2N^2}{K^*(K^*+1)} < R < \frac{2N^2}{K^*\ (K^*-1)} \qquad (3.8)$$

where,

$$R = (\ \frac{\sum_{j=1}^{n} D_j h_{jo}}{M_o}\ ) \qquad (3.9)$$

For $N = N^*$ to be optimal the total annual cost, $TC(N^*)$ must satisfy the following conditions,

$$TC(N^*) < TC(N^*+ 1) \qquad (3.10)$$

and $$TC(N^*) < TC(N^*- 1) \qquad (3.11)$$

Table 1 gives the range of values of (3.8) for different values of N and K. In practice K is generally less than 4. The procedure to find the optimal annual variable cost is presented in the form of an algorithm.

The algorithm:

1) Calculate,

$$R = \frac{\sum_{j=1}^{m} D_j h_{jo}}{M_o}$$

2) Corresponding to a given value of N and the computed value of R, obtain the value of $K^*$ from Table 4.1.

3) Compute the total variable cost, TC(N),

$$TC(N) = (S_o + \frac{M_o}{K^*}) N + \frac{1}{2N} \sum_{j=1}^{m} [(h_{j1} + (K^*-1) h_{jo})D_j] \qquad (3.12)$$

4) If $N = N^*$ satisfies inequalities (3.10) and (3.11) then N is optimal. Stop. Otherwise, go to step 5.

5) If inequality (3.10) is violated then increase N by one. If inequality (3.11) is violated decrease N by one. In either case, repeat from step 2 till both the inequalities (3.10) and (3.11) are satisfied.

Example: Data for a problem involving the joint replenishment of four items is given in Table IV.

Table IV

| Item $j$ | $D_j$ | $h_{j1}$ | $h_{jo}$ |
|---|---|---|---|
| 1 | 10,000 | 0.21 | 0.16 |
| 2 | 8,000 | 0.25 | 0.18 |
| 3 | 6,000 | 0.18 | 0.11 |
| 4 | 5,000 | 0.30 | 0.20 |

The single order cost for the customer and supplier are

$$S_o = 19 \quad \text{and} \quad M_o = 33$$

Individual Ordering Policy:

The optimal time interval for the customer is

$$t^* = \left[ \frac{2S_o}{\sum_{j=1}^{m} h_{j1} D_j} \right]^{\frac{1}{2}} = 0.079$$

and the variable cost per year is calculated as,

$$V(C(t^*)) = (2S_o (\sum_{j=1}^{4} h_{j1} D_j))^{\frac{1}{2}} = 480.66$$

To find $K^*$, calculating

$$\frac{2M_o}{t^{*2} (\sum_{j=1}^{4} h_{jo} D_j)} = 2.456$$

From inequality (3.4) we find that $K^* = 2$ satisfies the condition. The variable cost of supplier becomes,

$$V(S(K^*t^*)) = \frac{M_o}{K^*t^*} + \frac{(K^*-1)\ t^*}{2} \left( \sum_{j=1}^{4} D_j\ h_{jo} \right) = 378.71$$

Thus the total variable cost of the individual ordering policy = 480.66 + 378.71 = 859.37.

Integrated Ordering Policy:

For this policy, refering to Eq. (3.7) the variable cost per year is minimized for K = 1 and the value is

$$V(C(t^*),\ S(K^*t^*)) = 795.18.$$

The corresponding optimal time interval of ordering for customer is determined from Eq. (3.6). The $t^*$ value is 0.13. The net savings by integrated ordering policy as compared to individual ordering policy is 64.19 and percentage saving is of the order of 8 percent.

Now let us consider the optimization of integrated ordering policy with respect to $N^*$. We know that,

$$R = = \frac{\sum_{j=1}^{4} D_j\ h_{jo}}{M_o} = 130.3$$

Corresponding to N and R, the value of K is found from Table 4.1. The variable cost per year is computed from Eq. (3.12). The results are presented in a tabular form below:

| N | K* | V(C(N), S(N/K*)) |
|---|---|---|
| 6 | 1 | 818.6 |
| 7 | 1 | 798.3 |
| 8 | 1 | 796.0 |
| 9 | 1 | 805.0 |
| 10 | 1 | 824.0 |

The optimal value of $N = N^*$ is equal to 8 and the total variable cost per year corresponding to this value of N is 796.

Case II: Separate Ordering Cost for Each Item.

In this case we consider that each item has its own ordering cost and no cost reduction is possible by joint replenishment of different items. For any item 'j', the annual variable, $C_j(t_j, K_j)$, is given by,

$$C_j(t_j, K_j) = (S_j + \frac{M_j}{K_j}) \frac{1}{t_j} + \frac{1}{2} D_j \, (h_{j1} + (K_j - 1) \, h_{jo}) \, t_j \qquad (3.13)$$

where $K_j$ is a positive integer. Further,

$K_j = 1$ implies that the j-th item is ordered by the supplier at every $t_j$ interval, where $t_j$ is the customer cycle time for the j-th item,

$K_j = 2$ implies that the j-th item is ordered by the supplier at every $2t_j$ interval;

$K_j = 3$ implies that the j-th item is ordered by the supplier at every $3t_j$ interval and so on.

In Eq. (3.13), $t_j$ is a continuous variable and the optimal value of $t_j = t_j^*$ is given by,

$$t_j^* = \left[ \frac{2(S_j + M_j/K_j)}{D_j(h_{j1} + (K_j - 1)\ h_{jo}} \right]^{\frac{1}{2}} \qquad (3.14)$$

The total annual variable cost of the j-th item is determined by substituting the expression for $t_j^*$ in Eq. (3.13). Simplification leads to

$$C_j(t_j^*, K_j) = (2(S_j + M_j/K_j)(h_{j1} + (K_j-1)\ h_{jo})D_j)^{\frac{1}{2}} \quad (3.15)$$

In Eq. (3.15) the only unknown variable is $K_j$. Various values of $K_j$ are substituted in the Eq. (3.15) viz., $K_j = 1,2,3,\ldots$ . The $K_j = K_j^*$ is so chosen that the annual variable cost of the j-th item given by Eq. (3.15) is minimized. Alternatively, the following inequality can be used to determine the optimal value of $K_j = K_j^*$,

$$K_j^* \ (K_j^* + 1) > \frac{M_j}{S_j} \ \left( \frac{h_{j1} - h_{jo}}{h_{jo}} \right) \geq (K_j^* - 1)\ K_j^* \qquad (3.16)$$

Considering all the m items, the total annual variable cost is given as,

$$C = \sum_{j=1}^{m} C_j \ (t_j^*, K_j^*) \qquad (3.17)$$

When $h_{j1} \geq h_{jo}$, the inequality (3.16) gives the value of $K_j^*$. However, for $h_{jo} \geq h_{j1}$, the inequality (3.16) yields negative value of $K_j^*$ which is meaningless and $K_j^*$ is assigned a value equal to one.

Individual Ordering Policy:

When the supplier and the customer operate independently the total annual variable cost, $C_j(t_j)$, of the j-th item for the customer can be expressed as

$$C_j(t_j) = \frac{S_j}{t_j} + \frac{1}{2} D_j\ t_j\ h_{j1} \tag{3.18}$$

The optimization of Eq. (3.18) with respect to $t_j$ results in the following classical EOI formula

$$t_j^* = \left(\frac{2S_j}{D_j h_{j1}}\right)^{\frac{1}{2}} \tag{3.19}$$

Substituting $t_j^*$ in Eq. (3.18) gives

$$C_j(t_j^*) = (2S_j\ D_j h_{j1})^{\frac{1}{2}} \tag{3.20}$$

The supplier receives orders for the j-th item after every $t_j^*$ interval and determines the optimal reorder interval for the given $t_j^*$. The supplier's annual variable cost for the j-th item, $S_j\ (t_j^*,\ K_j)$, becomes

$$S_j(t_j^*,\ K_j) = \frac{M_j}{K_j\ t_j^*} + \frac{(K_j-1)\ t_j^*}{2}\ D_j\ h_{jo} \tag{3.21}$$

The following inequality can be used to determine the optimal value of $K_J = K_J^*$,

$$K_J^* (K_J^* + 1) > \frac{2M_J}{t_J^{*2} \, D_J \, h_{Jo}} \geq K_J^* (K_J^* - 1)$$

or

$$K_J^* (K_J^* + 1) > \left(\frac{M_J}{S_J}\right) \left(\frac{h_{J1}}{h_{Jo}}\right) \geq K_J^* (K_J^* - 1) \qquad (3.22)$$

Substituting the value of $K_J^*$ in Eq. (3.21), we get the optimal value of annual variable cost of the supplier for given $t_J^*$.

Example. Data for four items are given below in Table V.

Table V

| Item J | $D_J$ | $h_{J1}$ | $h_{Jo}$ | $S_J$ | $M_J$ |
|---|---|---|---|---|---|
| 1 | 8940 | 0.16 | 0.2 | 25 | 16 |
| 2 | 3260 | 0.2 | 0.15 | 15 | 20 |
| 3 | 1532 | 0.05 | 0.03 | 7 | 12 |
| 4 | 734 | 0.6 | 0.4 | 10 | 40 |

Integrated Ordering Policy:

The value of $K_J^*$ corresponding to each item can be obtained from inequality (3.16). For those items which have $h_{Jo} > h_{J1}$, the value of $K_J^*$ is equal to one. Substituting the value of $K_J^*$ in Eq. (3.15) we obtain the total cost for

each item. The values obtained for $K_j^*$ and $C_j^*(t_j^*, K_j^*)$ for each item are tabulated below:

| Item j | $(M_j/S_j)(h_{j1}-h_{jo}/h_{jo})$ | $K_j^*$ | $t_j^*$ | $C_j^*(t_j^*, K_j^*)$ |
|---|---|---|---|---|
| 1 | - | 1 | 0.24 | 342.50 |
| 2 | 0.44 | 1 | 0.33 | 213.60 |
| 3 | 1.14 | 1 | 0.70 | 54.00 |
| 4 | 2.00 | 2 | 0.48 | 209.90 |

From the last column, we find that the total cost for all the items, C(N) is 820.

Individual Ordering Policy:

The $t_j^*$ interval and the annual variable cost of the customer, $C_j(t_j^*)$ are obtained from Eqs. (3.19) and (3.20), respectively. The value of $K_j^*$ for each item for the supplier is computed from (3.22). Substituting $K_j^*$ in (3.21) we get the annual variable cost for the supplier. Results are presented below in a tabular form:

| Item j | $C_j(t_j^*)$ | $\frac{M_j}{S_j}(h_{j1}/h_{jo})$ | $K_j^*$ | $S_j(t_j^*, K_j^*)$ | $t_j^*$ |
|---|---|---|---|---|---|
| 1 | 267.43 | 0.512 | 1 | 85.56 | 0.187 |
| 2 | 139.85 | 1.77 | 1 | 95.24 | 0.21 |
| 3 | 32.74 | 2.86 | 2 | 23.83 | 0.43 |
| 4 | 93.85 | 6 | 3 | 125.15 | 0.21 |

The total variable cost by individual ordering policy is 863.65. Comparing the total variable costs of the individual and integrated ordering policies yields a saving of 5.3 percent for the integrated ordering policy.

Case III: Separate Order Cost Alongwith Joint Order Cost:

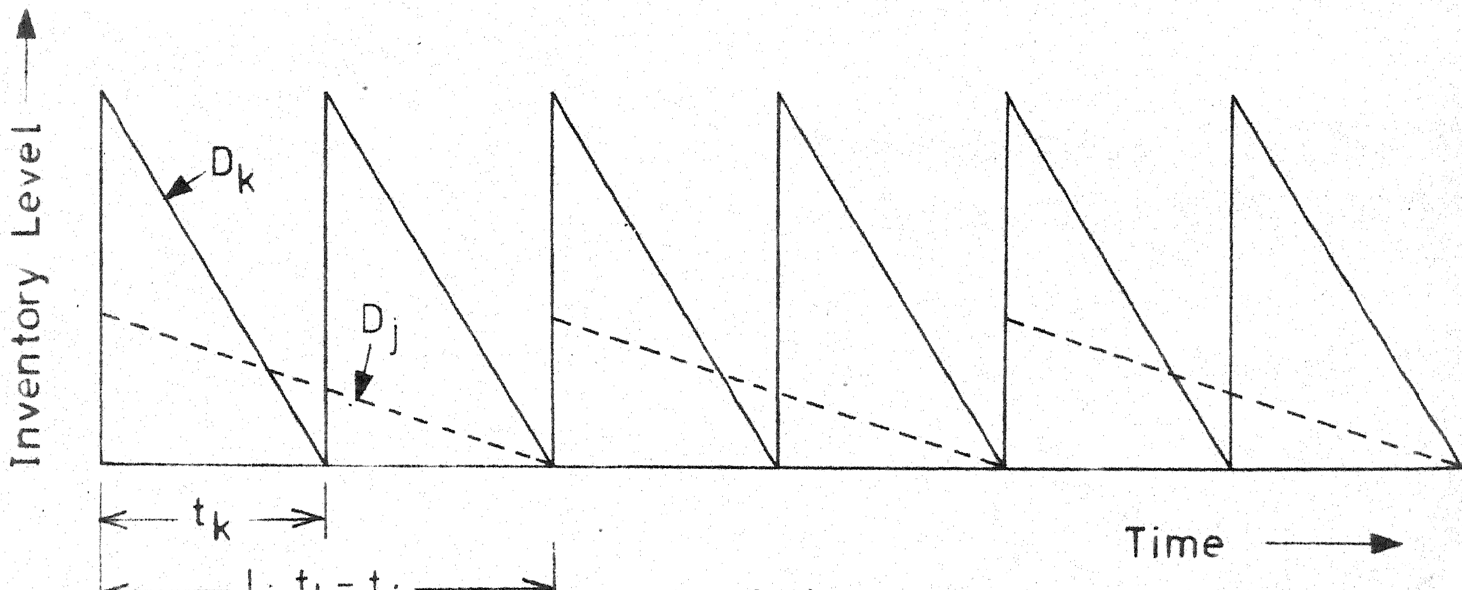
In this case we consider a practical situation of a single supplier-single customer inventory system for multiple items. Let N be the integer number of replenishments per year of the fastest moving item at the customer stocking point. Further, let the items be indexed in descending order of the frequency of ordering at the customer stocking point. The item which is replenished N times per year is indexed as 1.

The order cost structure under consideration is such that it has a constant part, i.e., the order cost component of which does not vary with the types of items ordered and a variable part which depends upon the types of items ordered. Mathematically, the annual order cost for the customer, $O_C$, can be expressed as,

$$O_C = S_o \cdot N + \sum_{j=1}^{m} S_j \left(\frac{N}{L_j}\right)$$

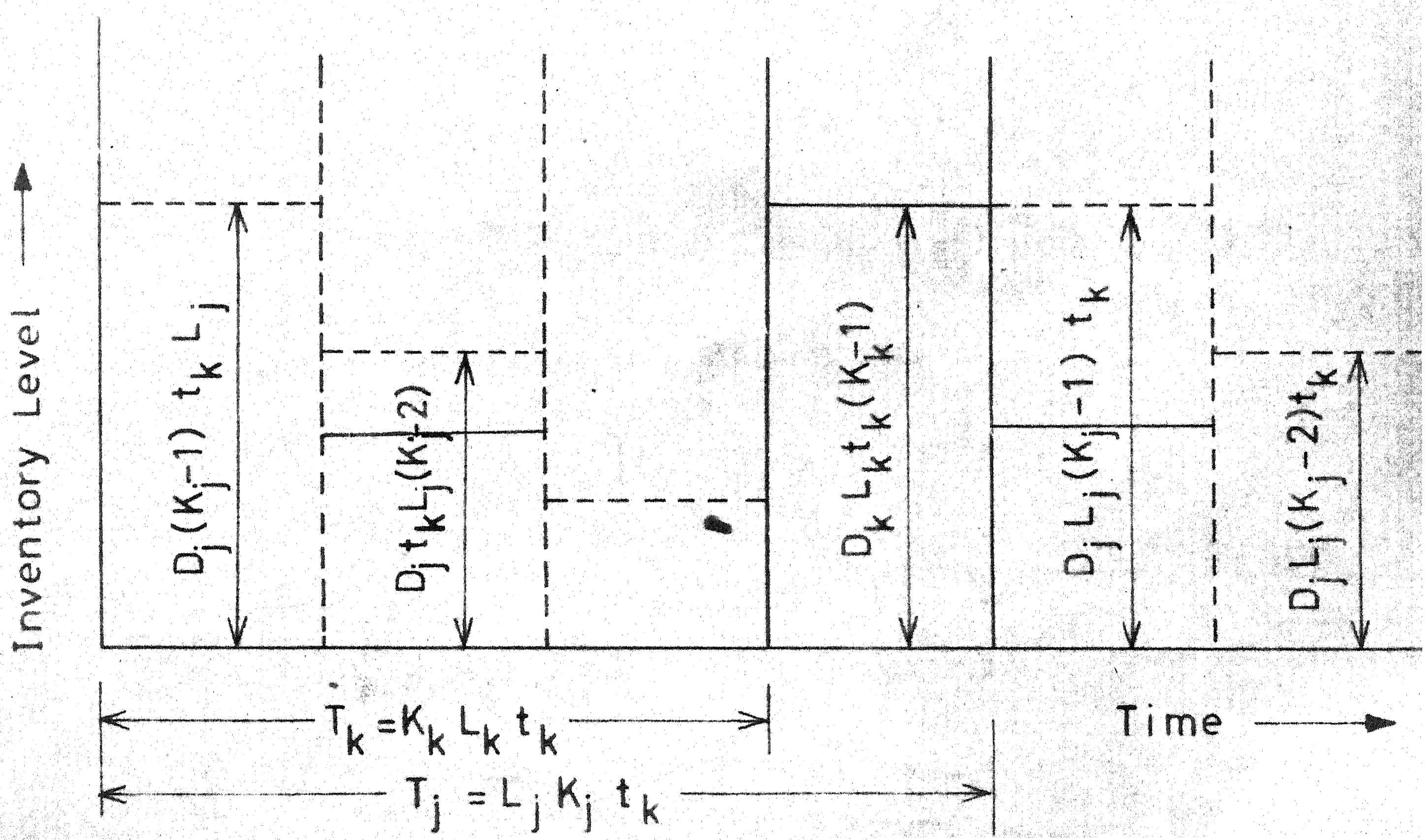
where $L_j$ is a positive integer and

$L_j=1$ implies that the customer orders the j-th item every cycle



Inventory Level
$D_k$
$D_j$
$t_k$
$L_j \; t_k = t_j$
Time


(a)



Inventory Level
$D_j(K_j-1)\; t_k L_j$
$D_j t_k L_j (K_j-2)$
$D_k L_k t_k (K_k-1)$
$D_j L_j (K_j-1)\; t_k$
$D_j L_j (K_j-2) t_k$
$T_k = K_k L_k t_k$
$T_j = L_j K_j \; t_k$
Time


(b)

$L_j=2$ implies that the customer orders the j-th item every alternate cycle.

$L_j=3$ implies that the customer orders the j-th item every third cycle and so on.

Similarly the order cost per year for the supplier, $O_S$, can be written as,

$$O_S = M_o \cdot N_o + \sum_{j=1}^{m} M_j \left(\frac{N}{L_j \cdot K_j}\right)$$

where $N_o$ is the number of orders placed by the supplier per year and is dependent on N, $K_j$ and $L_j$; j = 1,2,..., m.

The total order cost per year, $O_T$, is given by

$$O_T = O_C + O_S = (S_o \cdot N + \sum_{j=1}^{m} \left(\frac{N}{L_j}\right) S_j)$$

$$+ (M_o \cdot N_o + \sum_{j=1}^{m} M_j \left(\frac{N}{L_j \cdot K_j}\right))$$

$$= (S_o \cdot N + M_o \cdot N_o) + (\sum_{j=1}^{m} (S_j + \frac{M_j}{K_j}) \frac{N}{L_j}) \qquad (3.23)$$

The annual variable cost for the j-th item is

$$C_j(N, L_j, K_j) = (S_j \left(\frac{N}{L_j}\right) + M_j \left(\frac{N}{K_j \cdot L_j}\right))$$

$$+ \frac{1}{2} (h_{j1} + (K_j-1) h_{jo}) \frac{D_j \cdot L_j}{N} \qquad (3.24)$$

and the total cost per year, C(N), is given by

$$C(N) = (S_o.N + M_o.N_o) + \sum_{J=1}^{m} ((S_J + \frac{M_J}{K_J}) \frac{N}{L_J} + \frac{1}{2} \frac{D_J.L_J}{N} (h_{J1} + (K_J-1)\, h_{Jo}))$$

$$= (S_o.N + M_o.N_o) + \sum_{J=1}^{m} C_J (N, L_J, K_J) \tag{3.25}$$

We choose particular values of $L_J = L_J^*$ and $K_J = K_J^*$ such that,

$$C_J (N, L_J^*, K_J^*) < C_J (N, L_J^*, (K_J^* - 1)), \quad \text{and}$$

$$C_J (N, L_J^*, K_J^*) < C_J (N, L_J^*, (K_J^* + 1)), \quad \text{and}$$

$$C_J (N, L_J^*, K_J^*) < C_J (N, (L_J^*-1), K_J^*), \quad \text{and}$$

$$C_J (N, L_J^*, K_J^*) < C_J (N, (L_J^*+1), K_J^*).$$

Simplifying the above conditions we get,

$$\frac{2N^2}{L_J^* (L_J^* + 1)} < P_J < \frac{2N^2}{L_J^* (L_J^* - 1)}, \quad \text{and} \tag{3.26}$$

$$\frac{2N^2}{L_J^{*2} (K_J^* + 1)\, K_J^*} < R_J < \frac{2N^2}{L_J^{*2} (K_J^* - 1)\, K_J^*} \tag{3.27}$$

where,

$$P_J = \frac{(h_{J1} + (K_J^* - 1)\, h_{Jo})\, D_J}{(S_J + M_J/K_J^*)} \tag{3.28}$$

$$R_J = \frac{D_J\, h_{Jo}}{M_j} \tag{3.29}$$

From (3.26), we get,

$$L_j^* (L_j^* + 1) > \frac{2N^2}{P_j} > L_j^* (L_j^* - 1) \qquad (3.30)$$

In tables 1,2 and 3, the values of N, $L_j^*$ and $K_j^*$ are presented for a range of values of $R_j$ and $P_j$. For given values of N, $R_j$ and $P_j$ the values of $L_j^*$ and $K_j^*$ are obtained by referring to these tables. In practice both $L_j^*$ and $K_j^*$ are less than or equal to 3, i.e., the maximum value of $(L_j.K_j)$ that any item 'j' can have is 9.

Since $N_o$, the number of orders placed by the supplier per year, is dependent on N, $L_j$ and $K_j$, we outline a procedure of choosing the best value of $N_o$. We may recall that the items are arranged in descending order of their ordering frequencies and therefore for the first item, $L_1 = 1$. Further, for the first item the ordering interval of the supplier $(L_1 . K_1)$ will vary from one to three as $1 \leq K_1 \leq 3$. Obviously, the first item will play a dominant role in deciding the value of $N_o$. The various steps involved in choosing a particular value of $N_o$ are:

1) Find $L_j$, $K_j$ values by minimizing $C_j(N, L_j, K_j)$ for all items, $j = 1,2,\ldots, m$. Compute C(N) from Eq. (3.25).

2) List all items which have $L_1^* = 1$. The item which has the minimum value of $(L_1^* \; K_1^*)$ is indexed as the first item with $L_1^* = L_1^*$ and $K_1^* = K_1^*$.

3) If $(L_i.K_i)$ is not equal to 3 go to step 8. Else reduce $(L_1.K_1)$ to 2, where $K_1$ represents all items having $L_1.K_1 = 3$, $i = 1,2,\ldots, m$, and check if $N_o$ is reduced. If yes, then compute C(N) and let this be referred as C1(N) and go to step 4. Otherwise, retain previous values of $N_o$, $L_j$ and $K_j$.

4) Check if $(L_j.K_j) = 2$ for any other item $j = 2,\ldots, m$. Else go to step 6. Change all items having $(L_j.K_j) = 2$ value to $(L_j'.K_j') = 3$ and compute C(N). Let this C(N) be referred as C2(N). Calculate $\Delta C1(N) = (C(N)-C1(N))$ and $\Delta C2(N) = (C(N) -C2(N))$. If both $\Delta C1(N)$ and $\Delta C2(N)$ are negative then retain $L_j$, $K_j$ and $N_o$ values of step 1 and stop; otherwise, go to step 5.

5) If $\Delta C1(N)$ is greater than $\Delta C2(N)$ then assign $N_o$, $L_j$ and $K_j$ values of step 3 and stop; otherwise, assign $N_o$, $L_j$ and $K_j$ values of step 4 and stop.

6) If $(L_j.K_j)$ is not equal to 2 for any item, check if $(L_j.K_j) = 4$ for $j = 2,\ldots, m$. Change all $(L_j.K_j) = 4$ to $(L_j'.K_j') = 3$ by either making $L_j' = 1$ and $K_j' = 3$ or $L_j' = 3$ and $K_j' = 1$ depending on whichever results in lesser penalty cost. Computer C(N) and refer it as C3(N). Calculate $\Delta C1(N) = (C(N) - C1(N))$ and $\Delta C3(N) = (C(N) - C3(N))$. If both $\Delta C1(N)$ and $\Delta C3(N)$ are negative, then retain $L_j$, $K_j$ and $N_o$ values of step 1 and stop; otherwise, go to step 7.

7) If $\Delta C1(N) > \Delta C3(N)$, assign $L_j, K_j$ and $N_o$ values of step 3 and stop; otherwise, assign $L_j$, $K_j$ and $N_o$ values of step 6 and stop.

8) If $(L_1.K_1)$ is equal to one, increase $K_j$ by one for all items having $(L_j.K_j) = 1$. Compute $C(N)$ and call it $C4(N)$. If $C4(N) < C(N)$ go to Step 9; otherwise, retain values of $L_j$, $K_j$ and $N_o$ of step 1 and stop.

9) If $(L_1.K_1) \neq 1$ or $(L_1.K_1') = 2$ then

i) Change all items having $(L_1.K_1) = 2$ to $(L_1.K_1'') = 3$ and compute $C(N)$. Let this be referred as $C5(N)$.

ii) Change all items having $(L_1.K_1) = 2$ and $(L_1.K_1) = 4$ to $(L_1.K_1'') = 3$ and compute $C(N)$. Label it as $C6(N)$.

iii) Change all items having $(L_j.K_j) = 3$ to $(L_j'.K_j') = 2$. Compute $C(N)$. Label it as $C7(N)$.

iv) Change all items having $(L_j.K_j) = 3$ to $(L_j'.K_j') = 2$ and $(L_j.K_j) = 9$ to $(L_j'.K_j') = 6$. Compute $C(N)$. Label it $C8(N)$. Let $C'(N) = \min(C5(N), C6(N), C7(N), C8(N))$. If $C4(N)$ has been found, calculate $\Delta C'(N) = (C4(N) - C'(N))$. If $\Delta C'(N)$ is negative adopt the values of step 8. Else go to step 10. If $C4(N)$ has not been found. Calculate $\Delta C'(N) = (C(N) - C'(N))$. If $\Delta C'(N)$ is negative retain $L_j$, $K_j$ and $N_o$ values of step 1 and stop; otherwise go to step 10.

10) Assign $L_j$, $K_j$ and $N_o$ values corresponding to $C'(N)$ and stop.

We next discuss an algorithm developed for the determination of the variable cost per year.

The Algorithm:

1) For each item, calculate,

$$R_j = \frac{D_j\, h_{jo}}{M_j} \text{ , and}$$

$$P_j = \frac{(h_{j1} + (K_j - 1)\, h_{jo})\, D_j}{(S_j + M_j/K_j)} \text{ , } K_j = 1, 2, 3, \ldots$$

2) Corresponding to each value of $P_j$ find the value of $L_j$ from inequality (3.30). For this value of $L_j$ look into the respective table for given N and $R_j$ to obtain $K_j$. If the value of $K_j$ is the same as used to obtain the particular $P_j$, then $(L_j^*, K_j^*)$ is a possible ordering policy; otherwise, look for $K_j$ corresponding to the next value of $L_j$.

3) Similarly obtain all ordering policies for different values of $L_j$.

4) Compute the costs corresponding to each ordering policy and adopt the one which results in minimum cost.

5) Obtain the value of $N_o = N_o^*$ by the procedure outlined earlier, which also involves the computation of $C(N)$ for $N_o$.

6) The optimal value of $N = N^*$ if the following conditions are satisfied,

$$C(N^*) < C(N^*+1), \quad \text{and} \tag{3.31}$$

$$C(N\ ) < C(N^*-1) \tag{3.32}$$

If either one of these conditions is violated, proceed to Step 7. Else stop.

7) If inequality (3.31) is violated increase N by one and if inequality (3.32) is violated reduce N by one and repeat from step 2.

An Alternative Approach:

The best values of $L_j$ and $K_j$ can also be obtained graphically by having a plot of

i) N versus $\dfrac{2N^2}{L_j^{*\,2}(K_j^* + 1)\ K_j^*}$ and

ii) N versus $\dfrac{2N^2}{L_j^*\ (L_j^* + 1)}$

as shown in Fig. 3.3. This figure has been drawn for $L_j = 1,2,3$, and $K_j = 1,2,3$ and if needed curves corresponding to higher values of $L_j$ and $K_j$ can be drawn. In this figure the curves are labelled as $L_j = 1,2,3$ and $K_j = (i, p)$, where $i = 1,2,3$ denotes the value of $K_j$ corresponding to $L_j = p$, $p = 1,2,3$. The graphical method involves the following steps for the determination of $L_j^*$ and $K_j^*$.

1) For any item j, draw horizontal and vertical lines for $Y_o = R_j$ and $X_o = N$.

2) Let $K_j = (i, p)$ correspond to the curve just below, the point $(X_o, Y_o)$ for various values of $p = 1, 2, 3$. Then $K_j = i$ for that value of $p$.

3) Plot $P_j$ for $K_j = (i, p)$ by drawing a horizontal line $Y_p = P_j$. If $(X_o, Y_p)$ lies between curves $L_j = p$ and $L_j = (p-1)$, then $L_j = p$ and $K_j = i$, is a feasible ordering policy.

Example: Data for five items are given in Table VI below:

Table VI

| Item | $D_j$ | $S_j$ | $M_j$ | $h_{j1}$ | $h_{jo}$ |
|---|---|---|---|---|---|
| 1 | 10,000 | 8 | 15.0 | 0.21 | 0.16 |
| 2 | 8,000 | 8 | 13.5 | 0.25 | 0.18 |
| 3 | 6,000 | 7.5 | 13.5 | 0.18 | 0.11 |
| 4 | 3,000 | 7 | 13.0 | 0.15 | 0.1 |
| 5 | 1,000 | 7 | 12.0 | 0.3 | 0.2 |

The joint order cost for customer and supplier are:

$S_o = 11$ and $M_o = 15$.

Solution:

Integrated Approach:

Step 1: Calculate $R_j$ and $P_j$ for each item. The results are given below:

| Item $J$ | $R_J$ | $P_J$ | | |
|---|---|---|---|---|
| | | $K_J = 1$ | $K_J = 2$ | $K_J = 3$ |
| 1 | 106.6 | 91.33 | 238.70 | 407.60 |
| 2 | 106.6 | 93.00 | 233.00 | 390.00 |
| 3 | 48.8 | 51.4 | 122.00 | 200.00 |
| 4 | 23.0 | 22.4 | 55.50 | 92.64 |
| 5 | 16.67 | 15.8 | 38.46 | 63.60 |

Step 2: Obtain feasible ordering policies $(L_J, K_J)$ from the tables for different values of N, $R_J$ and $P_J$ as shown below;

| N | $K_1^*$ | $L_1^*$ | $K_2^*$ | $L_2^*$ | $K_3^*$ | $L_3^*$ | $K_4^*$ | $L_4^*$ | $K_5^*$ | $L_5^*$ | No |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 3 | 7 |
| 8 | 1 | 1 | 1 | 1 | $2,\underline{1}$ | $1,\underline{2}$ | 1 | 2 | 1 | 3 | 8 |
| 9 | 1 | 1 | 1 | 1 | $2,\underline{1}$ | $1,\underline{2}$ | $1,\underline{3}$ | $3,\underline{1}$ | $2,\underline{1}$ | $2,\underline{3}$ | 9 |

Step 3: For item 3 we have 2 feasible ordering policies, viz., (2,1) and (1,2) for N = 8 and N = 9. We adopt policy (2,1) as it yields lower cost. Similarly for item 4 and 5 we have alternate feasible ordering policies. The policies which result in lower costs are underlined as shown in the above table.

Step 4: Calculate the total cost per year for different values of N.

| N | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $S_o.N+M_o.N_o$ | Total cost C(N) |
|---|---|---|---|---|---|---|---|
| 7 | 311 | 293.35 | 224.03 | 148.07 | 108.6 | 182 | 1267.05 |
| 8 | 315.5 | 297 | 219 | 108.125 | 106.91 | 208 | 1254.535 |
| 9 | 323.6 | 304.61 | 214.5 | 135 | 107 | 234 | 1318.71 |

Step 5: Since for items 1 and 2, $(L_1^*.K_1^*) = 1$, increase $K_1^*$ to 2 for $i = 1,2$, and compute $C(N) = C5(N)$.

| N | $N_o$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $S_o.N+M_o.N_o$ | Total cos C5(N) |
|---|---|---|---|---|---|---|---|---|
| 7 | 5 | 372.8 | 348.9 | 224.03 | 148.07 | 108.6 | 152 | 1354.4 |
| 8 | 5 | 355.25 | 333 | 219 | 108.13 | 106.91 | 163 | 1285.29 |
| 9 | 6 | 345 | 323.86 | 214.5 | 135 | 107 | 189 | 1314.36 |

On comparing C5(N) and C(N) for various values of N, we find that for N = 9 the total cost in step 5 has reduced.

Step 6: Since for N = 9 the total cost C5(N) has reduced, further checking is done by effecting the following changes in the ordering policy corresponding to N = 9.

i) Change $(L_1^*.K_1')$ from 2 to $(L_1^*.K_1'') = 3$ by increasing $K_1^*$ by one for i = 1,2 and change $(L_3^*.K_3^*) = 2$ to $(L_3'.K_3^*) = 3$ by increasing $L_3^*$ by one.

ii) Change $(L_j^*.K_j^*) = 3$ for j = 4,5 to $(L_j'.K_j') = 2$ for j = 4,5. The corresponding total costs C6(N) for these 2 policies are

| Policy | N | $N_o$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $S_o.N + M_o.N_o$ | Total cost C(N) |
|---|---|---|---|---|---|---|---|---|---|
| (i) | 9 | 3 | 41.44 | 383.6 | 342 | 135 | 107 | 144 | 1424.04 |
| (ii) | 9 | 5 | 345 | 323.86 | 214.5 | 140 | 118.83 | 174 | 1316.19 |

As these 2 ordering policy variations for N = 9 result in increase of total cost, the optimal policy for N = 9 is $L_i^* = 1$, $K_1^* = 2$ for i = 1,2, $L_3^* = 1$, $K_3^* = 1$, $L_4^* = 1$, $K_4^* = 3$ and $L_5^* = 3$, $K_5^* = 1$ and the corresponding value of C5(N) is 1314.36. However, the overall optimal policy corresponds to $N^* = 8$ and the total cost is 1254.54.

Individual Ordering Policy

Step 1: Calculate $R_j$ for all items as shown below:

| Item i | $R_j$ |
|---|---|
| 1 | 262.5 |
| 2 | 250 |
| 3 | 144 |
| 4 | 64.2 |
| 5 | 42.85 |

Step 2: From Table 1, obtain $L_j^*$. With this value of $L_j^*$ compute the annual variable cost of the j-th item and the total cost, $C_1(N)$, for the customer as given below:

$$C_j(N, L_j^*) = S_j\left(\frac{N}{L_j^*}\right) + \frac{1}{2} D_j h_{j1} \frac{L_j^*}{N} \quad \text{and} \tag{3.33}$$

$$C_1(N) = S_o \cdot N + \sum_{j=1}^{m} C_j (N, L_j^*) \tag{3.34}$$

Results are given in a tabular form below:

| N | $L_1^*$ | $L_2^*$ | $L_3^*$ | $L_4^*$ | $L_5^*$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $S_o \cdot N$ | Total cost $= C_1(N)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 1 | 1 | 1 | 2 | 195 | 189 | 128 | 84 | 66 | 88 | 750 |
| 9 | 1 | 1 | 1 | 2 | 2 | 189 | 183 | 128 | 81 | 65 | 99 | 745 |
| 10 | 1 | 1 | 1 | 2 | 2 | 185 | 180 | 129 | 80 | 65 | 110 | 749 |
| 11 | 1 | 1 | 1 | 2 | 2 | 183 | 179 | 129 | 80 | 66 | 121 | 759 |

The results indicate that the best policy for the customer is to have 9 periods i.e., $N = N^* = 9$.

Step 3: Corresponding to $N^* = 9$ the suppliers $K_j'$'s can be obtained from the inequality (3.5) given below:

$$K_j^* (K_j^* + 1) > \frac{2N^2 \quad M_j}{L_j^{*2} \; D_j \; h_{jo}} \geq K_j^* (K_j^* - 1) \tag{3.5}$$

Thus the values of $K_j$'s are $K_1^* = 1$, $K_2^* = 1$, $K_3^* = 2$, $K_4^* = 1$, $K_5^* = 2$ and $N_o^* = 9$. The corresponding costs are as shown:

| $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $M_o.N_o^*$ | Total Cost $C_2(N)$ |
|---|---|---|---|---|---|---|
| 135 | 121.5 | 97.42 | 58.5 | 49.22 | 135 | 596.64 |

The total cost of individual ordering, $C_3(N) = C_1(N) + C_2(N)$

$$= 745 + 596.64 = 1341.64.$$

Percentage saving by integrated approach

$$= \frac{87.105}{1254.535} = 6.9 \text{ percent}$$

3.6 DISCUSSION OF RESULTS AND CERTAIN FEATURES OF THE MODELS:

From the numerical examples given alongwith each case it becomes immediately evident that integrated ordering policy results in savings as compared to an independent ordering policy of the supplier and the customer. Moreover, it can be seen that the application of the various algorithms is very simple and is even amenable to hand calculations for medium sized problems. Certain special features of the models which come forth intuitively are worth mentioning. They are:

1. If the (M/S) ratio is quite small and $(h_{jo}/h_{j1})$ very high, then supplier will not store any item and will supply as and when he receives an order. The model reduces to an on-order inventory system.

2. If the (M/S) ratio is very high and $(h_{jo}/h_{j1})$ very low, then supplier cycle time will be much larger than customer cycle time, i.e. $K^*$ is large.

3) If $(h_{jo}/h_{jl}) \rightarrow 0$, then supplier warehouse may be thought of as having an infinite capacity, i.e., all items are stocked by the supplier all the time. In such a case the results of independent and integrated ordering policies will be very close to each other.

4) In an integrated system the customer may or may not operate at its optimal level. The penalty cost incurred by the customer is more than offset by the savings of the supplier, thus minimizing the system cost. If the supplier and the customer do not belong to the same system, then the customer has to be compensated for the extra cost incurred by him by adopting the integrated ordering policy. In the example solved the customer incurred a cost equal to 796.41 by the integrated ordering policy and 745.0 by the individual ordering policy. Thus the extra cost incurred by the customer is 51.41. The savings of the supplier by the integrated model was 138.51. Hence even if the supplier compensates the customer for the extra cost incurred by him, the net savings are 87.10, i.e. 6.9 percent of the total system cost.

5) In an integrated system the supplier cycle time is less than or equal to the supplier cycle time in an individual ordering policy.

6) The computational effort increases with increase in disparity among the various items as regards their parameters.

# CHAPTER IV

## SINGLE SUPPLIER - m CUSTOMER INVENTORY SYSTEM

In this chapter we deal with the single supplier - m customer inventory system. Problem formulation and solution methodology for obtaining near optimal solution is provided alongwith the relevant literature.

### 4.1 LITERATURE REVIEW

The only published article on single supplier - m customer inventory system is by Schwarz [8]. He has considered a continuous review inventory system involving deterministic parameters. For this problem he has stated that the optimal policy observes certain properties which are briefly given below:

1. Replenishment of the item is made by the supplier only when the supplier as well as at least one of the customers have zero inventory.
2. Deliveries are made to any given customer only when that customer has zero inventory.
3. All deliveries made to any given customer between successive deliveries to the supplier are of equal size.

Schwarz has proposed a heuristic for a special single cycling policy where all the customers have zero inventory

when replenishment is made by the supplier. He has compared the results with certain analytical bounds and found on the basis of these tests that the heuristic yields near optimal results.

## 4.2 STATEMENT OF THE PROBLEM.

Consider a single item inventory system comprising of one supplier and m customers. The customers obtain their supplies from the supplier who in turn procures the supplies from some other source.

The objective is to evolve decision rules for the supplier and the customers as to how much and how often to place an Order such that total cost of the inventory system is minimized over the planning horizon.

## 4.3 ASSUMPTIONS

The system characteristics are as follows:

1. Planning horizon is finite.
2. Demand rate of the item at each customer stocking point is deterministic and static.
3. Shortages are not allowed.
4. Replenishments cannot be split.
5. Quantity discounts are not permitted.
6. Holding cost is linear in nature.
7. Lead time is zero for both the supplier and the customers.

## 4.4 NOMENCLATURE

The notations which have been used throughout this chapter are given below:

$m$ number of customers

$N$ number of periods in the planning horizon

<u>For the j-th Customer</u>

$D_j$ - demand for the planning horizon

$S_j$ - separate order cost component

$t_j$ - time interval between successive orders

$h_j$ - stock holding cost per unit per unit time

$L_j$ - a positive integer, whose value gives the number of reference intervals at which the j-th customer places the order

<u>For the supplier:</u>

$M_o$ - single order cost component

$h_o$ - stock holding cost per unit per unit time

$N_o$ - no. of orders placed in a planning horizon

$T_j$ - time interval between successive orders of the supplier for the j-th customer

$K_j$ - a positive integer whose value when multiplied to $L_j$ gives the interval at which the supplier orders for the j-th customer

$[a/b]$ - maximum integer value (0, x), i.e., for negative values of $(a/b)$ take the value of zero.

$(S/n_a, n_b, \ldots, C/L_j)$ - signifies an ordering policy when the supplier places orders every $n_a, n_b, \ldots$ intervals and the j-th customer orders every $L_j$ periods. At any period no more than one order is placed by the supplier. However, the suppliers order size need not be the same.

## 4.5 PROBLEM FORMULATION AND SOLUTION METHODOLOGY

The total cost of the system, $C(N)$, comprises of the cost incurred by the supplier and all the customers. Mathematically,

$$C(N) = M_o.N_o + \sum_{j=1}^{m} C_j(N, L_j) + \sum_{j=1}^{m} H_s(j) \qquad (4.1)$$

where $H_s(j)$ is the cost of holding the item at the supplier point for supplying the demand of the j-th customer and $C_j(N, L_j)$ is the total cost corresponding to the j-th customer. $C_j(N, L_j)$ is given by

$$C_j(N, L_j) = S_j \left(\frac{N}{L_j}\right) + \frac{1}{2} h_j.D_j \frac{L_j}{N} \qquad (4.2)$$

For the minimization of $C_j(N, L_j)$ Goyal's technique [3] is adopted to choose a particular value of $L_j = L_j'$. Depending upon the value of $L_j'$, the customers are classified as very high, high, medium,... frequency customers. The

lower the value of $L_j'$ higher is the frequency of ordering. This yields the following inequality

$$\frac{2N^2}{L_j'(L_j'+1)} < R_j < \frac{2N^2}{L_j'(L_j'-1)} \tag{4.3}$$

where,

$$R_j = \frac{h_j \cdot D_j}{S_j} \tag{4.4}$$

Goyal [3] has constructed a table which gives ranges of $R_j$ for various values of N and $L_j'$, $L_j' = 1,2,3$. As part of this work, Goyal's table has been extended to include $L_j' = 4$ and is presented in Table 4.1. Given N and $R_j$ the value of $L_j'$ can be obtained from this table.

The time interval after which the supplier places the order signifies the policy of the supplier. The supplier incurs ordering cost at all such intervals irrespective of the fact whether there is an order from the customer or not. For a specified suppliers ordering policy the objective of the supplier is to minimize the overall cost of the inventory system as given by eqn. (4.2). Various preferred ordering policies of the supplier are evaluated and the one which yields the least cost is selected. The various ordering policies of the supplier are analyzed below.

Policy - I: Supplier Orders Every Period:

For this ordering policy of the supplier, the total number of orders placed by the supplier in the planning horizon is equal to N. Depending on the ordering policy of the customers, the supplier order size may vary from period to period. At the first period, the supplier order size will account for one order from all the customers. For any other period the order quantity will be effected by the ordering policy of the various customers. Let us consider an example. Assume that the customers have been classified into three groups having $L_J' = 1, 2,$ and 3. Fig. 4.1 shows the order size of the supplier at various discrete points in time (beginning of the period) during the planning horizon. The circles and the crosses indicate the points at which the supplier and the customers place orders. The size of the vertical bar signifies the amount ordered by the supplier for the set of customers ordering at that instant. The suppliers order size for the first period is the sum of the orders of all the customers. In the second period the order size corresponds to customers having $L_J' = 1$. For the third period order size corresponds to the set of customer having $L_J' = 1$ and 3. In this way the order size for any period can be determined.

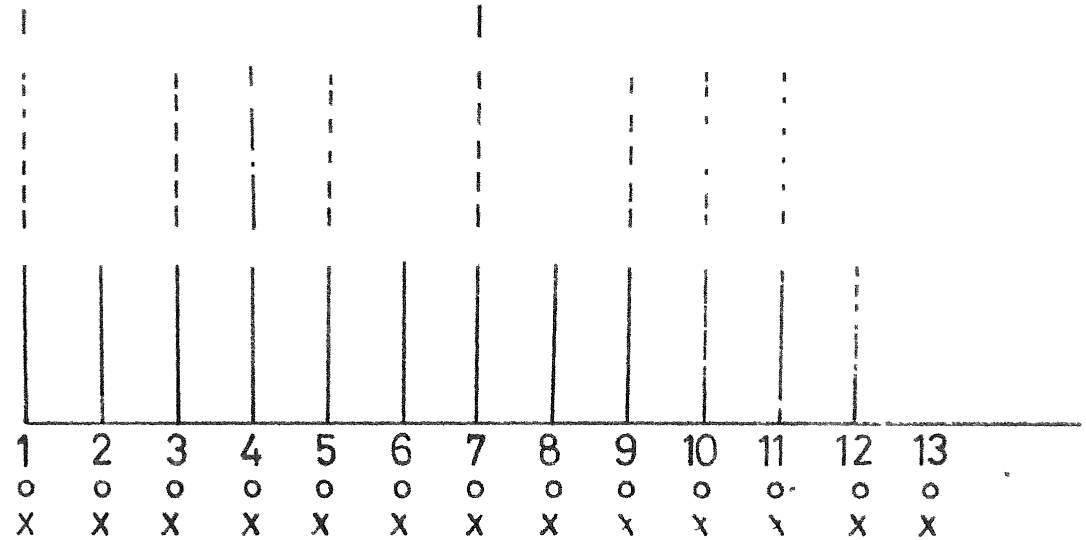



Fig. 4·1 Ordering schedule of supplier & customers

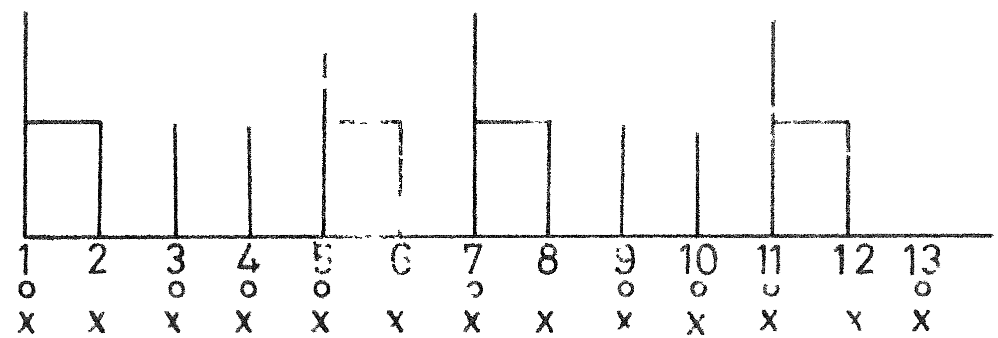



Fig. 4·2 Ordering schedule of supplier & customers with inventory level of supplier for policy (S/2, 3; C/1)

Considering assumption 7, the holding cost of the supplier becomes zero, i.e., $H_s(j) = 0$, $j = 1,2,\ldots m$. The total cost of the system, $C(N)$, is

$$C(N) = M_o N_o + \sum_{j=1}^{m} C_j(N, L_j')$$

$$= M_o.N + \sum_{j=1}^{m} [S_j(N/L_j') + \frac{1}{2} D_j h_j \frac{L_j'}{N}] \qquad (4.5)$$

Policy II : Supplier Orders Every Second and Third Period

For policy 2, $N_o$ can be expressed as,

$$N_o = \lfloor \frac{N}{2} \rfloor + \lfloor \frac{N-3}{6} \rfloor \qquad (4.6)$$

We now consider various cases for $L_j' = 1,2,3$ and 4.

Case 1 : $L_j' = 1$

For this case the ordering policy is given by (S/2,3;C/1). Fig. 4.2 shows the ordering schedule of the supplier and customers as well as the amount of inventory held at the supplier stocking point. The holding cost for the supplier is

$$H_s(j) = \begin{cases} (2x+1) \dfrac{D_j h_o}{N^2}, & 6x + 2 \leq N < 6(x+1) \\ (2x) \dfrac{D_j h_o}{N^2}, & 6x \leq N \leq 6x+1 \end{cases} \qquad (4.7)$$

where $x = 0,1,2,\ldots$ . As regards $C_j$ $(N, L_j')$, component of the

total cost is given by eq. (4.2). It is observed that it is same as in policy 1.

Case 2: $L_j' = 2,3$ and 4:

For customers having $L_j' = 2,3$, and 4, the various ordering policies are (S/2,3;C/2), (S/2,3;C/3) and (S/2,3;C/4). No holding cost is incurred by the supplier. $C_j(N, L_j')$ can be calculated from eq. (4.2).

Policy III Supplier Orders Every Alternate Period:

The number of orders placed by the supplier is given by,

$$N_o = \lfloor N/2 \rfloor \tag{4.8}$$

Under this policy there are various cases corresponding to $L_j' = 1,2,3$ and 4. These cases are discussed next.

Case 1 $L_j' = 1$

The ordering policy can be represented by (S/2; C/1). Fig. 4.3(a) shows the ordering scheduling of the supplier and the customers as well as the inventory held by the supplier during various periods. The holding cost expression is

$$H(j) = x \cdot \frac{D_j h_o}{N^2}, \quad 2x \leq N \leq 2x + 1 \tag{4.9}$$

where $x = 0,1,2,\ldots$ .

Case 2: $L_j' = 2$

The ordering policy is (S/2;C/2) for which supplier incurs no holding cost.

Case 3: $L_j' = 3$

The ordering policy is (S/2; C/3). For this policy the supplier incurs holding cost as shown in Fig. 4.3(b). The holding cost can be calculated as

$$H_s(j) = \begin{cases} 0 , & N < 4 \\ (x+1)\dfrac{3D_j h_o}{N^2} , & 6(x+1) \leq N \leq 6(x+1)+3 \\ (3x+1)\dfrac{D_j h_o}{N^2} , & N = 6x+4 \\ (3x+2)\dfrac{D_j h_o}{N^2} , & N = 6x+5 \end{cases} \qquad (4.10)$$

where $x = 0,1,2,\ldots$ .

Case IV: $L_j' = 4$:

The ordering policy is (S/4; C/4) for which no holding cost is incurred by the supplier.

For each of the above four cases, $C_j(N, L_j')$ is calculated from eq. (4.2).

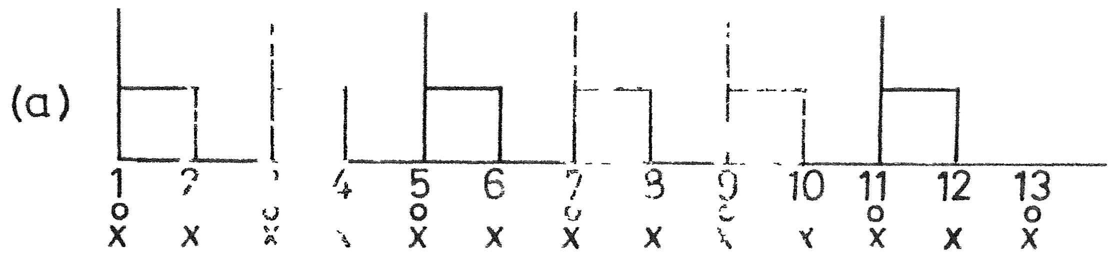



Fig. 4·3 Ordering schedule of supplier & customers with inventor level of supplier for (a) Policy (S/2 ; C/1)
(b) Policy (S/2 ; C/3)

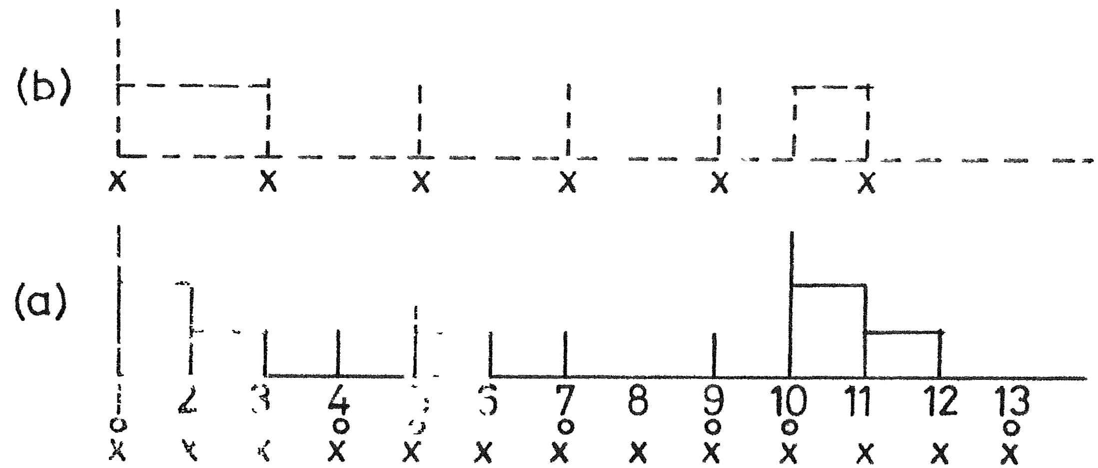



Fig. 4·4 Ordering schedule of supplier & customers with inventory level for (a) Policy (S/3, 4 ; C/1)
(b) Policy (S/3, 4 ; C/2)

Policy IV : <u>Supplier Orders Every Third And Fourth Period</u>:

$N_o$ is given by the expression,

$$N_o = \lfloor N/4 \rfloor + \lfloor \frac{N-3}{3} \rfloor - \lfloor \frac{N-12}{12} \rfloor \tag{4.11}$$

Case 1. $f_j' = 1$

The ordering policy is (S/3,4,, C/1). Fig. 4.4(a) shows the ordering schedule of the supplier and the customers as well as the amount of inventory held during any period. The holding cost can be calculated as,

$$H_c(j) = \begin{cases} (8x) \dfrac{D_j h_o}{N^2}, & 12x \leq N \leq 12x+1 \\ (8x+1) \dfrac{D_j h_o}{N^2}, & N = 12x + 2 \\ (8x+3) \dfrac{D_j h_o}{N^2}, & 12x + 3 \leq N \leq 12x + 5 \\ (8x+4) \dfrac{D_j h_o}{N^2}, & 12x + 6 \leq N \leq 12x++ 7 \\ (8x+5) \dfrac{D_j h_o}{N^2}, & 12(x+1) - 4 \leq N \leq 12(x+1)-2 \\ (8x+6) \dfrac{D_j h_o}{N^2}, & N = 12(x+1) - 1 \end{cases} \tag{4.12}$$

where,

$x = 0,1,2,3,\ldots$ .

Case 2. $L_j^{'} = 2$

The ordering policy is (S/3,4; C/2). Fig. 4.4(b) shows the amount of inventory held at any period by the supplier. The holding cost of the supplier is

$$H_s(j) \begin{cases} (3x+2)\,\dfrac{2D_j\, h_o}{N^2}, & 12x+4 \le N \le 12(x+1) - 2 \\ (3x+1)\,\dfrac{2D_j\, h_o}{N^2}, & N = 12x + 3 \\ (3x)\,\dfrac{2D_j\, h_o}{N^2}, & 12x \le N \le 12x+2 \\ (3x+2.5)\,\dfrac{2D_j\, h_o}{N^2}, & N = 12(x+1) - 1 \end{cases} \tag{4.13}$$

where,

$x = 0,1,2,\ldots$ .

Case 3: $L_j^{'} = 3$ and 4

For customers having $L_j^{'} = 3$ and 4 the two ordering policies are,(S/3,4;C/3) and (S/3,4;C/4). No holding cost is incurred by the supplier.

For all the cases of this policy, $C_j(N, L_j^{'})$ is calculated from eq. (4.2).

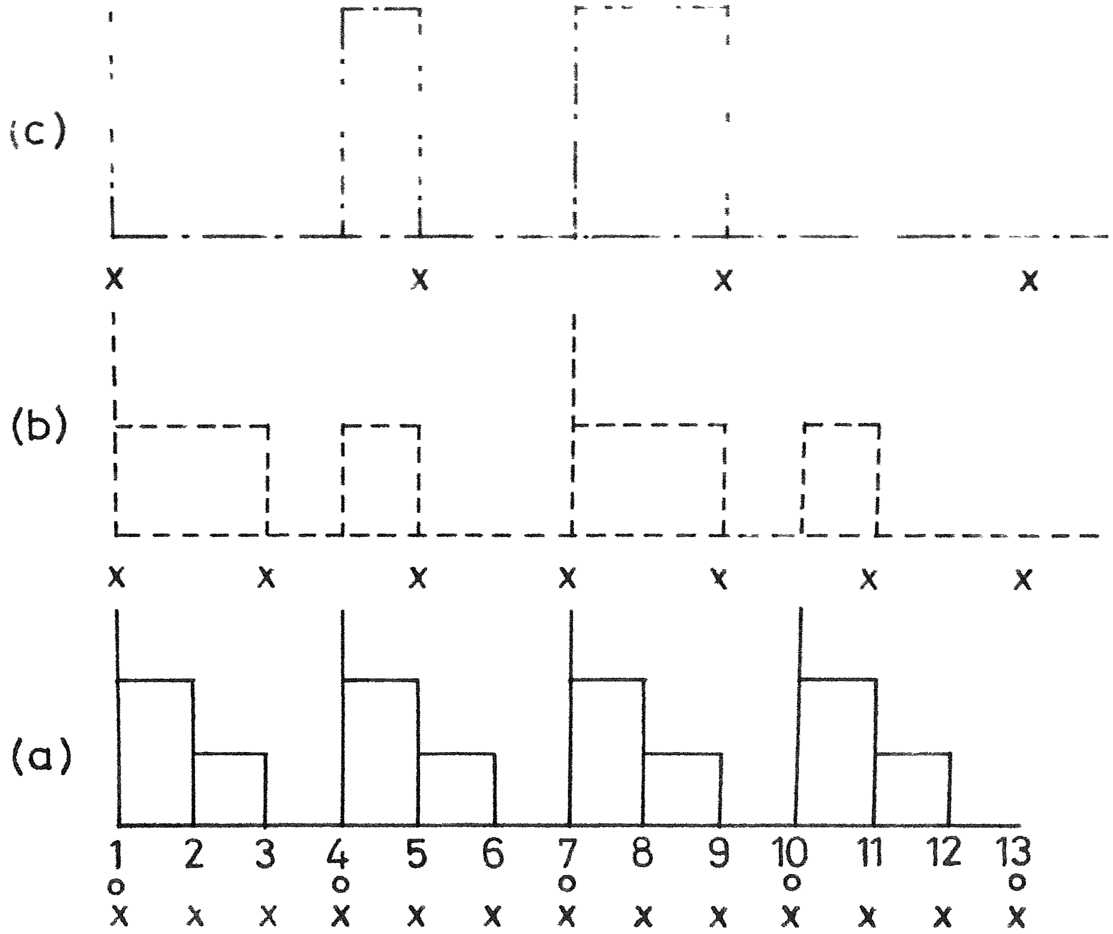



Fig. 4·5 Ordering schedule of supplier & customer
level of supplier for

Policy [illegible]. Supplier Places Orders Every Third Period:

The number of orders placed by the supplier is

$$N_o = \lfloor N/3 \rfloor$$

Cases corresponding to $L_J' = 1, 2, 3$ and 4 are discussed.

Case 1: $L_J' = 1$

The ordering policy is (S/3; C/1). Fig. 4.5(a) depicts the ordering schedule of the supplier and the customers as well as the amount of stock held by the supplier during any period. The holding cost is given by

$$H_S(J) = \begin{cases} [3(x+1)-2]\,\dfrac{D_J\, h_o}{N^2}, & N = 3(x+1)-1 \\[2ex] (3x)\,\dfrac{D_J\, h_o}{N^2}, & 3x \le N \le 3x+1 \end{cases}$$

where,

$x = 0, 1, 2, \ldots$ .

Case 2: $L_J' = 2$

The ordering policy is (S/3; C/2). Fig. 4.5(b) shows the inventory picture at the supplier stocking point. The holding cost of the supplier can be found from

$$H_S(J) = \begin{cases} (6x)\,\dfrac{2D_J h_o}{N^2}, & 12x-1 \le N \le 12x+2 \\[2ex] (6x+1)\,\dfrac{2D_J h_o}{N^2}, & N = 12x+3 \\[2ex] (6x+2)\,\dfrac{2D_J h_o}{N^2}, & N = 12x+4 \\[2ex] (6x+2.5)\,\dfrac{2D_J h_o}{N^2}, & N = 12x+5 \\[2ex] (6x+3)\,\dfrac{2D_J h_o}{N^2}, & 12x+6 \le N \le 12x+8 \\[2ex] (6x+4)\,\dfrac{2D_J h_o}{N^2}, & N = 12x+9 \\[2ex] (6x+5)\,\dfrac{2D_J h_o}{N^2}, & N = 12x+10 \end{cases}$$

where,

$x = 0, 1, 2, \ldots$ .

Case 3. $L'_J = 3$

The ordering policy is (S/3; C/3). For this policy supplier incurs no holding cost.

Case 4 $L'_J = 4$

The ordering policy is (S/3; C/4). The holding cost picture of the supplier is given in Fig. 4.5(c), The holding cost is given by

$$H_s(j) = \begin{cases} 12x \dfrac{D_j h_o}{N^2} & , 12x + 1 \leq N \leq 12x + 4 \text{ and } x = 0,1,2,\ldots \\ (12x+y) \dfrac{D_j h_o}{N^2} & , 12x + 4+y = N, x = 0,1,2,\ldots \text{and } y=1,2,3,4. \\ (12x+2z) \dfrac{D_j h_o}{N^2} & , 12x + 6+z = N, x = 0,1,2,\ldots \text{ and } z=3,4,5,6. \end{cases}$$

For all the cases $C_j(N, L_j')$ is calculated from eq. (4.2).

The supplier can proceed in a similar manner till $N_o = N/N = 1$. But in general for $N_o < \lfloor N/3 \rfloor$ the total cost of the system increases. Further search is justified only if $M_o$ is very high or $h_o$ is very small or both. It may be noted that during the development of the above policies the properties of an optimal ordering policy as proposed by Schwarz's have not been violated. Given below is an algorithm to obtain optimal (or near optimal) value of the total cost of the system.

<u>The Algorithm</u>:

1) Calculate $R_j$ for all the customers from eq. (4.4).

2) With given $R_j$ and $N$, find $L_j'$ from Table 4.

3) Consider policy I, Pick $N = N_1$ such that

$$C(N_1) < C(N_1+1), \text{ and}$$
$$C(N_1) < C(N_1-1).$$

4) Consider policy II.

a) For customers having $L_j' = 1$, compute $[H_s(j)+C_j(N,1)]$ corresponding to policy (S/2,3; C/1). Perturb $L_j'$ from 1 to 2. Compute $C_j(N, 2)$ from eq.(4.2) corresponding to policy (S/2,3; C/2). Pick min. $[H_s(j)+C_j(N,1), C_j(N,2)]$.

b) For other customers compute $C_j(N,L_j')$ from eq.(4.2). Compute the total cost of the system, $C(N)$, from eq.(4.1). Then choose $N = N_2$ such that

$$C(N_2) < C(N_2 \pm I), \qquad I = 1,2$$

Search for $N_2$ is restricted to the set $(N_1-2, N_1-1, N_1, N_1+1 \ldots)$.

5) Consider policy III.

a) For customers having $L_j' = 1$, compute $[H_s(j) + C_j(N,1)]$ for policy (S/2; C/1). Perturb $L_j' = 1$ to $L_j' = 2$ to obtain policy (S/2; C/2). Compute $C_j(N,2)$. Select min $[H_s(j) + C_j(N,1), C_j(N,2)]$

b) For customers having $L_j' = 2,4,6,\ldots$ the eq.(4.2) gives the value of $C_j(N, L_j')$ and $H_s(j) = 0$.

c) For customers having $L_j' = 3$, select min $[H_s(j) + C_j(N, 3), C_j(N, 2)]$ corresponding to policies (S/2; C/3) and (S/2; C/2) respectively. Compute $C(N)$. Choose that particular value of $N = N_3$ such that

$$C(N_3) < C(N_3 \pm I), \quad I = 1,2,$$

Search for $N_3$ is made from the set of N comprising of $(N_2-2,\ N_2-1,\ N_2,\ N_2+1,\ \ldots\ )$.

6) Consider policy IV.

a) For customers having $L_j' = 1$, select

$\min\ (H_s(j) + C_j(N, 1),\ H_s(j) + C_j(N, 2),\ C_j(N, 3))$

where $L_j'$ is perturbed to 2 and 3, respectively. The various policies are (S/3,4; C/1), (S/3,4; C/2) and (S/3,4; C/3).

b) For customers having $L_j' = 2$ select

$\min\ [H_s(j) + C_j(N,L_j'),\ C_j(N,\ L_j)]$

where $L_j'=3$ is obtained by perturbing $L_j'$. The two policies are (S/3,4; C/2) and (S/3,4;C/3).

c) Compute total system cost, C(N) from eq. (4.1). Choose $N = N_4$ such that

$$C(N_4) < C(N_4 \pm I) \quad , \quad I = 1,2,3.$$

7) Consider policy V.

a) For customers having $L_j' = 1$, consider the policies (S/3; C/1), (S/3; C/2) and (S/3; C/3) where $L_j'$ is perturbed to 2 and 3 for the second and third policy, respectively. Select min. $[C_j\ (N,\ L_j) + H_s(j)]$ for the 3 policies.

b) For customers having $L_j' = 2$, consider two policies (S/3; C/3) and (S/3; C/2). Pick the one which yields min. $(C_j(N,\ L_j') + H_s(j))$.

c) Compute C(N) from eq. (4.1). Select $N = N_5$ such that,

$$C(N_5) < C(N_5 \pm I), \quad I = 1,2,3.$$

$N_5$ belongs to the set $(N_4-2, N_4-1, N_4, N_4+1, \ldots)$

8) Then the optimal cost of the system for $N = N^*$ is given by

$$C(N^*) = \min((C(N_1), C(N_2), C(N_3), C(N_4), C(N_5))$$

Example: Sample data is given in Table 2 for a system having four customers. The single order cost for supplier, $M_o = 30$ and the stock holding cost for supplier/unit/unit time, $h_o =$ = 0.15

Table 2

| Item $j$ | $D_j$ | $S_j$ | $h_j$ |
|---|---|---|---|
| 1 | 10,000 | 8 | 0,16 |
| 2 | 8,000 | 8 | 0.11 |
| 3 | 6,000 | 15 | 0.08 |
| 4 | 1,000 | 12 | 0.2 |

Step 1: Calculate $R_j$ for each customer:

| Item $j$ | $R_j = D_j.h_j/S_j$ |
|---|---|
| 1 | 200 |
| 2 | 110 |
| 3 | 32 |
| 4 | 16.67 |

tep 2: Obtain the values of $L_j'$ from Table 1 corresponding o N and $R_j$.

| | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $L_1'$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| $L_2'$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $L_3'$ | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 5 |
| $L_4'$ | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 6 |

Step 3: Compute $C_j = (C_j(N, L_j') + H_s(j))$ for policy I. Results are shown below in a tabular form

| N | 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| $C_1$ | 232 | 200 | 181.3 | 170.3 |
| $C_2$ | 142 | 128 | 121.3 | 118.9 |
| $C_3$ | 120 | 123 | 125.0 | 121.1 |
| $C_4$ | 73 | 70 | 69.3 | 70.6 |
| $N_o$ | 4 | 5 | 6 | 7 |
| $M_o \cdot N_o$ | 120 | 150 | 180.0 | 210 |
| C(N) | 687 | 671 | 676.9 | 690.9 |

Step 4: Consider policy II and compute various costs. For customers with alternate policies the optimal policy is underlined.

| | N | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| $C_1$ | (S/2,3;C/2) | 340.0 | 290.7 | 256.6 | 232.0 | 213.8 | 200 |
| | (S/2,3;C/1) | 260.0 | 264.6 | 251.5 | 234.3 | 216.5 | 205 |
| $C_2$ | (S/2,3;C/2) | 196.0 | 170.7 | 153.7 | 142.0 | 133.8 | 128 |
| | (S/2,3;C/1) | 176.0 | 178.0 | 167.9 | 175.3 | 165.5 | 160 |
| $C_3$ | (S/2,3;C/2) | 123.0 | 125 | 121.1 | 120.0 | 120.8 | 122 |
| $C_4$ | (S/2,3;C/2) | 70.0 | 69.3 | 70.6 | – | – | – |
| | (S/2,3;C/3) | – | – | – | 69.5 | 69.3 | 70 |
| | $N_o$ | 4 | 4 | 5 | 5 | 6 | 7 |
| | $M_o . N_o$ | 120.0 | 120.0 | 150 | 150 | 180 | 210 |
| | C(N) | 749.0 | 749.6 | 726.8 | 713.5 | 717.7 | 730.0 |

optimal for this policy occurs at $N = N_2 = 8$ at $C(N_2) = 713.5$

Step 5: Consider policy III and compute the costs.

| | N | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|
| $C_1$ | (S/2;C/2) | 256.6 | 232.0 | 213.8 | 200 | 189.4 | 181.3 |
| | (S/2;C/1) | 262.1 | 257.8 | 235.0 | 235 | 222.6 | 226.2 |
| $C_2$ | (S/2;C/2) | 153.7 | 142.0 | 133.8 | 128 | 124.0 | 121.3 |
| | (S/2;C/1) | 192.4 | 194 | 180.6 | 184 | – | – |
| $C_3$ | (S/2;C/2) | 121.1 | 120.0 | 120.8 | 123 | 126.1 | 130.0 |
| | (S/2;C/3) | – | – | – | 158 | 152.7 | 157.5 |
| $C_4$ | (S/2;C/2) | 70.6 | 73.0 | 76.2 | 80 | – | – |
| | (S/2;C/3) | – | 78.7 | 74.9 | 76 | – | – |
| | (S/4;C/4) | – | – | – | – | 69.4 | 69.3 |
| $N_o$ | | 4 | 4 | 5 | 5 | 6 | 6 |
| $M_o \cdot N_o$ | | 120.0 | 120.0 | 150.0 | 150 | 180.0 | 180.0 |
| C(N) | | 722.0 | 687.0 | 693.3 | 677 | 688.9 | 681.9 |

Optimal occurs at $N = N_3 = 10$ at $C(N_3) = 677.0$

Step 6; Consider policy IV. Compute the various costs. For customers having $L_j' = 5, 6, 7, \ldots$ the $L_j'$ is so perturbed that minimum extra cost is incurred by the customer.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | (S/3,4;C/3) | 321.3 | 290.7 | 266.7 | 247.5 | 232.0 | 219.3 | 208.7 | 200.0 | 192.7 | 186.5 | 181.3 |
| | (S/3,4;C/1) | 281.2 | 253.5 | 235.0 | 235.1 | 246.0 | – | – | – | – | – | – |
| | (S/3,4;C/2) | 325.8 | 287.9 | 260.0 | 251.4 | 245.8 | 228.4 | 223.6 | 212.6 | 222.6 | 214.0 | 207.2 |
| 2 | (S/3,4;C/3) | 186.3 | 170.7 | 158.7 | 149.0 | 142.0 | 136.2 | 131.6 | 128.0 | 125.2 | 123.0 | 121.3 |
| | (S/3,4;C/1) | 212.8 | 195 | 184 | – | – | – | – | – | – | – | – |
| | (S/3,4;C/2) | 217.0 | 193.1 | 176.0 | 173.6 | 171.3 | 162.3 | 155.1 | 161.4 | – | – | – |
| 3 | (S/3,4;C/3) | 130 | 125 | 122 | 120.5 | 120.0 | 120.4 | – | – | – | – | – |
| | (S/3,4;C/2) | 176.3 | 165.2 | – | – | – | – | – | – | – | – | – |
| | (S/3,4;C/4) | – | – | – | – | – | – | 121.1 | 120.3 | 120.0 | 120.3 | 120.8 |
| 4 | (S/3,4;C/3) | 69.5 | 69.3 | 70.0 | – | – | – | – | – | – | – | – |
| | (S/3,4;C/4) | – | – | – | 69.4 | 69.3 | 69.8 | 70.5 | – | – | – | – |
| | (S/3,4;C/6) | – | – | – | – | – | – | – | 70.0 | 69.5 | 69.3 | 69.3 |
| $N_o$ | | 4 | 5 | 6 | 6 | 6 | 7 | 7 | 7 | 8 | 9 | 9 |
| $M_o \cdot N_o$ | | 120 | 150 | 180 | 180 | 180 | 210 | 210 | 210 | 240 | 270 | 270 |
| C(N) | | 787 | 768.5 | 765.7 | 754.3 | 743.3 | 755.7 | 742.0 | 730.0 | 741.4 | 769.1 | 762.7 |

Optimal lies at $N = N_4 = 15$ at $C(N_4) = 730.0$

Step 7: Consider policy V.

| | N | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|
| $C_1$ | (S/3;C/3) | 219.3 | 208.7 | 200 | 192.7 | 186.5 | 181.3 |
| | (S/3;C/2) | 281.6 | 262.1 | 254.0 | 257.8 | 250.3 | 239.0 |
| $C_2$ | (S/3;C/3) | 136.2 | 131.6 | 128.0 | 125.2 | 123.0 | 121.3 |
| | (S/3;C/2) | 204.9 | 192.4 | 193.4 | 194.0 | 194.5 | 183.9 |
| $C_3$ | (S/3;C/3) | 120.4 | 121.4 | 123.0 | 125.0 | 127.4 | 130 |
| | (S/3;C/4) | - | 176.2 | 168.3 | 162.2 | 160.8 | 159.7 |
| $C_4$ | (S/3;C/6) | 72.2 | 70.9 | 70.0 | 69.5 | 69.3 | 69.3 |
| $N_o$ | | 5 | 5 | 5 | 6 | 6 | 6 |
| $M_oN_o$ | | 150 | 150 | 150 | 180 | 180 | 180 |
| C(N) | | 682.1 | 682.6 | 671.0 | 692.4 | 686.2 | 681.9 |

Optimal lies at $N = N_5 = 15$ at $C(N_5) = 671.0$

This ordering policy is identical to the ordering policy $N_1$ with system cost $C(N_1)$. The only difference is that each period is further sub-divided into three parts.

Step 8: The system optimal lies at $N = N^* = 15$ and the corresponding cost of the system,

$$C(N^*) = 671.0$$

Step 9: Compute Schwarz's lower bound when all the customers and the supplier operate at their optimal. Total cost is given by

$$C_1(N) = [2M_o\ h_o\ D_o]^{\frac{1}{2}} + \sum_{j=1}^{m} (2S_j\ h_j\ D_j)^{\frac{1}{2}} \qquad (4.14)$$

where $$D_o = \sum_{j=1}^{m} D_j = 25000 \qquad (4.15)$$

$\therefore \quad C_1(N) = 474.3 + 468.0 = 942.3$

Savings by integrated model as compared to Schwarz's lower bound = 942.3 - 671.0 = 271.3 and percentage saving is of the order of 40.4 percent.

## 4.6 DISCUSSION OF RESULTS:

The proposed algorithm yields optimal or near optimal solution. The results shows a marked improvement when compared with Schwarz's lower bound where supplier and all the customers are allowed to operate independently. Depending on the values of $L_j'$ for all the customers a particular policy may be more suitable than the others to obtain the best results. As compared to the previous algorithms this algorithm is amenable to hand calculation for small problems only. When the disparity among the various customers increase as regards their parameters the computational effort is considerably increased.

# CHAPTER V

## CONCLUSION AND SCOPE FOR FURTHER WORK

In the preceding chapters we have presented a number of models for inventory problems normally encountered in any practical distribution network. For each of the models integrated ordering policy as well as individual ordering policy of the supplier and the customers have been considered. For the individual ordering policy closed form solutions have been obtained. However, for the integrated ordering policy iterative solution methodologies which lead to optimal or near optimal solutions have been developed. Most of the models have been tested with numerical examples to illustrate the solution methodologies. The examples show that considerable savings accrue in case of integrated ordering policy as compared to the conventional individual ordering policy. The proposed algorithms are computationally efficient and for reasonable size problems hand calculations are adequate. However, for larger sized problems one may have to resort to the use of computer.

### 5.1 SCOPE FOR FURTHER RESEARCH:

The present work on the analysis of distribution networks is still far from complete. During the course of

this study number of avenues for further research emerged. The main directions in which the present work can be extended are:

1. Development of optimal ordering policy for single supplier - m customer inventory system dealing in single and multiple items.

2. Carrying out the sensitivity analysis for the 2-echelon inventory system for both the single item as well as the multi-item case. Development of analytical expressions for the savings accruing under integrated ordering policy as compared to the individual ordering policy.

3. Development of optimal ordering policy for a multi-echelon inventory system having deterministic and/or stochastic parameters.

4. Development of models to account for inflationary environment.

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Table 1: Range of values of $R_j$ and $P_j$ for given values of N and $K_j^* \cdot L_j^* = 1$.

| N | Range of $P_j$ | Range of $R_j$ | | |
|---|---|---|---|---|
| | | $K_j^* = 1$ | $K_j^* = 2$ | $K_j^* = 3$ |
| 1 | 1 - ∞ | 1 - ∞ | 0.33 - 1 | 0.16 - 0.33 |
| 2 | 4 - ∞ | 4 - ∞ | 1.33 - 4 | 0.66 - 1.33 |
| 3 | 9 - ∞ | 9 - ∞ | 3.00 - 9 | 1.50 - 3.00 |
| 4 | 16 - ∞ | 16 - ∞ | 5.33 - 16 | 2.66 - 5.33 |
| 5 | 25 - ∞ | 25 - ∞ | 8.33 - 25 | 4.16 - 8.33 |
| 6 | 36 - ∞ | 36 - ∞ | 12.00 -- 36 | 6.00 - 12.00 |
| 7 | 49 - ∞ | 49 - ∞ | 16.33 - 49 | 8.16 - 16.33 |
| 8 | 64 - ∞ | 64 - ∞ | 21.33 - 64 | 10.60 - 21.33 |
| 9 | 81 - ∞ | 81 - ∞ | 27.00 - 81 | 13.50 - 27.00 |
| 10 | 100 - ∞ | 100 - ∞ | 33.30 - 100 | 16.60 - 33.30 |
| 11 | 121 - ∞ | 121 - ∞ | 40.33 - 121 | 20.10 - 40.33 |
| 12 | 144 - ∞ | 144 - ∞ | 48.00 - 144 | 24.00 - 48.00 |
| 13 | 169 - ∞ | 169 - ∞ | 56.33 - 169 | 28.10 - 56.33 |
| 14 | 196 - ∞ | 196 - ∞ | 65.33 - 196 | 32.67 - 65.33 |
| 15 | 225 - ∞ | 225 - ∞ | 75.00 - 225 | 37.50 - 75.00 |
| 16 | 256 - ∞ | 256 - ∞ | 85.33 - 256 | 42.67 - 85.33 |
| 17 | 289 - ∞ | 289 - ∞ | 96.33 - 289 | 48.16 - 96.33 |
| 18 | 324 - ∞ | 324 - ∞ | 108.00 - 324 | 54.00 - 108.00 |
| 19 | 361 - ∞ | 361 - ∞ | 120.33 - 361 | 60.16 - 120.33 |
| 20 | 400 - ∞ | 400 - ∞ | 133.33 - 400 | 66.60 - 133.33 |

Table 2: Range of values of $R_j$ and $P_j$ for given values of N and $K_j^* \cdot L_j^* = 2$.

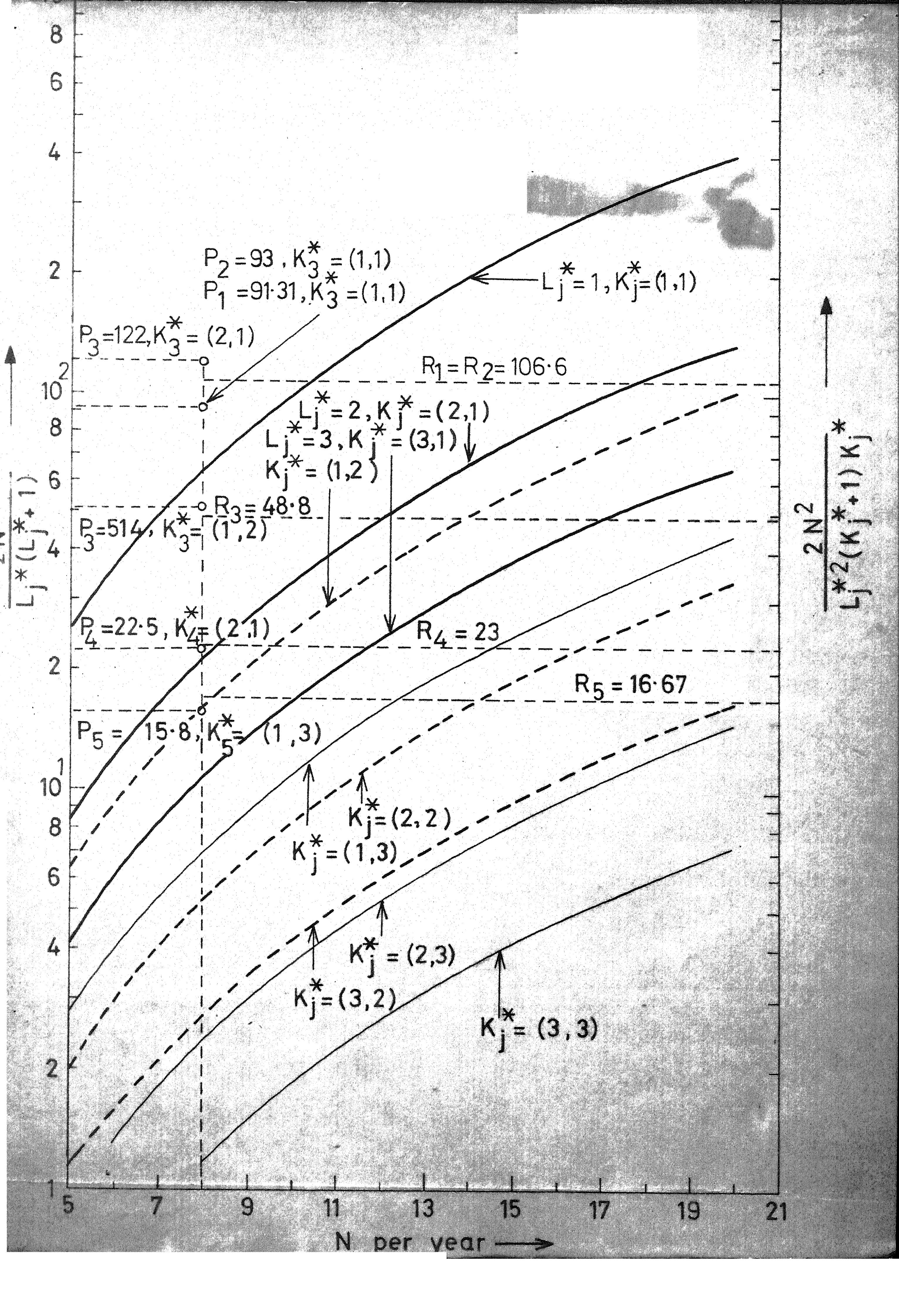
| N | Range of $P_j$ | Range of $R_j$ | | |
|---|---|---|---|---|
| | | $K_j^* = 1$ | $K_j^* = 2$ | $K_j^* = 3$ |
| 1 | 0.33 - 1 | 25 - ∞ | 0.0833 - 0.25 | 0.04165 - 0.0833 |
| 2 | 1.33 - 4 | 1 - ∞ | 0.33 - 1 | 0.165 - 0.33 |
| 3 | 3 - 9 | 2.25 - ∞ | 0.75 - 2.25 | 0.375 - 0.75 |
| 4 | 5.33 - 16 | 4 - ∞ | 1.33 - 4 | 0.665 - 1.33 |
| 5 | 8.33 - 25 | 6.25 - ∞ | 2.08 - 6.25 | 1.04 - 2.08 |
| 6 | 12 - 36 | 9 - ∞ | 3 - 9 | 1.5 - 3 |
| 7 | 16.33 - 49 | 12.25 - ∞ | 4.08 -12.25 | 2.04 - 4.08 |
| 8 | 21.33 - 64 | 16 - ∞ | 5.33 -16 | 2.665 - 5.33 |
| 9 | 27 - 81 | 20.25 - ∞ | 6.75 -20.25 | 3.375 - 6.75 |
| 10 | 33.3 - 100 | 25 - ∞ | 8.33 -25 | 4.165 - 8.33 |
| 11 | 40.33 - 121 | 30.25 - ∞ | 10.08 -30.25 | 5.04 -10.08 |
| 12 | 48.00 - 144 | 36 - ∞ | 12 -36 | 6 -12 |
| 13 | 56.33 - 169 | 42.25 - ∞ | 14.08 -42.25 | 7.04 -14.08 |
| 14 | 65.33 - 196 | 49 - ∞ | 16.33 -49 | 8.165 -16.33 |
| 15 | 75.00 - 225 | 56.25 - ∞ | 18.75 -56.25 | 9.375 -18.75 |
| 16 | 85.33 - 256 | 64 - ∞ | 21.33 -64 | 10.665 -21.33 |
| 17 | 96.33 - 289 | 72.25 - ∞ | 24.08 -72.25 | 12.04 -24.08 |
| 18 | 108.0 - 324 | 81 - ∞ | 27 -81 | 13.5 -27 |
| 19 | 120.33 - 361 | 90.25 - ∞ | 30.08 -90.25 | 15.04 -30.08 |
| 20 | 133.33 - 400 | 100. - ∞ | 33.3 -100 | 16.665 -33.3 |

Table 3: Range of values of $R_j$ and $P_j$ for given values of N, $K_j^* \cdot L_j^* = 3$.

| N | Range of $P_j$ | Range of $R_j$ | | |
|---|---|---|---|---|
| | | $K_j^* = 1$ | $K_j^* = 2$ | $K_j^* = 3$ |
| 1 | 0.33 - 0.16 | 0.111 - ∞ | 0.037 - 0.111 | 0.0185 - 0.037 |
| 2 | 1.33 - 0.66 | 0.44 - ∞ | 0.148 - 0.44 | 0.074 - 0.148 |
| 3 | 3.00 - 1.50 | 1 - ∞ | 0.33 - 1 | 0.166 - 0.33 |
| 4 | 5.33 - 2.66 | 1.77 - ∞ | 0.59 - 1.77 | 0.295 - 0.59 |
| 5 | 8.33 - 4.16 | 2.77 - ∞ | 0.923 - 2.77 | 0.4615 - 0.923 |
| 6 | 12.00 - 6.00 | 4 - ∞ | 1.33 - 4 | 0.665 - 1.33 |
| 7 | 16.33 - 8.16 | 5.44 - ∞ | 1.813 - 5.44 | 0.9075 - 1.813 |
| 8 | 21.33 - 10.60 | 7.11 - ∞ | 2.37 - 7.11 | 1.185 - 2.37 |
| 9 | 27.00 - 13.50 | 9 - ∞ | 3.0 - 9.00 | 1.5 - 3.0 |
| 10 | 33.30 - 16.60 | 11.11 - ∞ | 3.703 - 11.11 | 1.851 - 3.703 |
| 11 | 40.33 - 20.10 | 13.44 - ∞ | 4.48 - 13.44 | 2.24 - 4.48 |
| 12 | 48.00 - 24.00 | 16 - ∞ | 5.33 - 16.0 | 2.665 - 5.33 |
| 13 | 56.33 - 28.10 | 18.77 - ∞ | 6.256 - 18.77 | 3.128 - 6.256 |
| 14 | 65.33 - 32.67 | 21.77 - ∞ | 7.256 - 21.77 | 3.628 - 7.256 |
| 15 | 75.00 - 37.50 | 25 - ∞ | 8.33 - 25.0 | 4.165 - 8.33 |
| 16 | 85.33 - 42.67 | 28.44 - ∞ | 9.48 - 28.44 | 4.74 - 9.48 |
| 17 | 96.33 - 48.16 | 32.11 - ∞ | 10.70 - 32.11 | 5.35 -10.70 |
| 18 | 108.00 - 54.00 | 36.00 - ∞ | 12.0 - 36.0 | 6.00 -12.00 |
| 19 | 120.33 - 60.16 | 40.11 - ∞ | 13.37 - 40.11 | 6.685 -13.37 |
| 20 | 133.33 - 66.6 | 44.44 - ∞ | 14.81 - 44.44 | 7.405 -14.81 |

Table 4.1: Range of values which $R_j$ can take for values of N and $L_j$.

| N | $L_j = 1$ | $L_j = 2$ | $L_j = 3$ | $L_j = 4$ |
|---|---|---|---|---|
| 1 | ∞ - 1 | 1 - 0.33 | 0.33 - 0.16 | 0.16 - 0.1 |
| 2 | ∞ - 4 | 4 - 1.33 | 1.33 - 0.66 | 0.66 - 0.4 |
| 3 | ∞ - 9 | 9 - 3.00 | 3.00 - 1.50 | 1.50 - 0.9 |
| 4 | ∞ - 16 | 16 - 5.33 | 5.33 - 2.66 | 2.60 - 1.6 |
| 5 | ∞ - 25 | 25 - 8.33 | 8.33 - 4.16 | 4.16 - 2.5 |
| 6 | ∞ - 36 | 36 - 12.00 | 12.00 - 6.00 | 6.00 - 3.6 |
| 7 | ∞ - 49 | 49 - 16.33 | 16.33 - 8.16 | 8.16 - 4.9 |
| 8 | ∞ - 64 | 64 - 21.33 | 21.33 - 10.60 | 10.60 - 6.4 |
| 9 | ∞ - 81 | 81 - 27.00 | 27.00 - 13.50 | 13.50 - 8.1 |
| 10 | ∞ - 100 | 100 - 33.30 | 33.30 - 16.60 | 16.60 - 10.0 |
| 11 | ∞ - 121 | 121 - 40.33 | 40.33 - 20.10 | 20.10 - 12.1 |
| 12 | ∞ - 144 | 144 - 48.00 | 48.00 - 24.00 | 24.00 - 14.4 |
| 13 | ∞ - 169 | 169 - 56.33 | 56.33 - 28.10 | 28.10 - 16.9 |
| 14 | ∞ - 196 | 196 - 65.33 | 65.33 - 32.67 | 32.67 - 19.6 |
| 15 | ∞ - 225 | 225 - 75.00 | 75.00 - 37.50 | 37.50 - 22.5 |
| 16 | ∞ - 256 | 256 - 85.33 | 85.33 - 42.67 | 42.67 - 25.6 |
| 17 | ∞ - 289 | 289 - 96.33 | 96.33 - 48.16 | 48.16 - 28.9 |
| 18 | ∞ - 324 | 324 -108.00 | 108.00 - 54.00 | 54.00 - 32.4 |
| 19 | ∞ - 361 | 361 -120.33 | 120.33 - 60.16 | 60.16 - 36.1 |
| 20 | ∞ - 400 | 400 -133.33 | 133.33 - 66.60 | 66.60 - 40.0 |



$P_2 = 93$ , $K_3^* = (1,1)$
$P_1 = 91{\cdot}31$, $K_3^* = (1,1)$
$L_j^* = 1, K_j^* = (1,1)$
$P_3 = 122, K_3^* = (2,1)$
$R_1 = R_2 = 106{\cdot}6$
$L_j^* = 2, K_j^* = (2,1)$
$L_j^* = 3, K_j^* = (3,1)$
$K_j^* = (1,2)$
$R_3 = 48{\cdot}8$
$P_3 = 514, K_3^* = (1,2)$
$P_4 = 22{\cdot}5, K_4^* = (2,1)$
$R_4 = 23$
$R_5 = 16{\cdot}67$
$P_5 = 15{\cdot}8, K_5^* = (1,3)$
$K_j^* = (2,2)$
$K_j^* = (1,3)$
$K_j^* = (2,3)$
$K_j^* = (3,2)$
$K_j^* = (3,3)$
$\frac{2N^2}{L_j^*(L_j^*+1)}$
$\frac{2N^2}{L_j^{*2}(K_j^*+1)K_j^*}$
N per year